(17) Canonical Transforms

To motivate our next theoretical step, canonical transformations, let's remind ourselves how we use Lagrangians and Hamiltonians to solve mechanics problems. I'll use the simple pendulum as a concrete example

How to do mechanics, step by step

1) Write T and U in Cartesian coordinates

$$T(\dot{\vec{r}}) = \frac{1}{2}m\left(\dot{x}^2 + \dot{y}^2\right) \quad , \quad U(\vec{r}) = mgy$$

2) Write transformation to generalized coordinates

$$\begin{aligned} \vec{r}(q) \text{ e.g. } x &= R\sin\phi \ , \ y = -R\cos\phi \\ \dot{\vec{r}}(q,\dot{q}) \text{ e.g. } \dot{x} &= R\cos\phi\,\dot{\phi} \ , \ \dot{y} = R\sin\phi\,\dot{\phi} \end{aligned}$$

3) Write $T(q,\dot{q})$ and $U(q)$

$$T\left(\phi,\dot{\phi}\right) = \frac{1}{2}mR^2\dot{\phi}^2 \quad , \quad U(\phi) = -mgR\cos\phi$$

4) Compute generalized momenta

$$p_i = \frac{\partial L}{\partial \dot{q}_i} \quad , \quad \text{with } L = T - U \quad \text{e.g. } p_\phi = mR^2\dot{\phi}$$

From here, you can continue on the Lagrangian path and...

Lagrangian

5) Compute $F_i = \frac{\partial L}{\partial q_i}$ e.g. $F_\phi = mgR\sin\phi$

6) Find equations of motion with $\dot{p}_i = F_i$ e.g. $\ddot{\phi} = \frac{g}{R}\sin\phi$

or you can use these generalized momenta in the Hamiltonian

Hamiltonian

5) Write $T(p,q)$ e.g. $T(p_\phi,\phi) = \frac{p_\phi^2}{2mR^2}$

6) Find equations of motion with $H = T + U$ and

$$\dot{q} = \frac{\partial H}{\partial p} \ , \ \dot{p} = -\frac{\partial H}{\partial q} \ \text{e.g.} \ \dot{\phi} = \frac{p_\phi}{mR^2} \ , \ \dot{p}_\phi = mgR\sin\phi$$

It may seem strange that we need to go through the Lagrangian to find the momenta used in the Hamiltonian, but this just highlights a difference between these two approaches.

The Lagrangian is based on a choice of generalized coordinates. Any choice will do, and the momenta are a result of that choice.

$$L'\left(Q,\dot{Q}\right) = L\left(q(Q),\dot{q}(Q,\dot{Q})\right)$$

for any $Q(q)$ transform (invertable, differentiable,...) e.g. from Cartesian to polar in 2D

$$\begin{aligned} Q(q) \ &\Rightarrow \ r = \sqrt{x^2+y^2} \ , \ \phi = \tan^{-1}\left(\frac{y}{x}\right) \\ &\Rightarrow \ L'\left(r,\phi,\dot{r},\dot{\phi}\right) = L\left(x(r,\phi),y(r,\phi),\dot{x}(...),\dot{y}(...)\right) \end{aligned}$$

Momenta result from our choice of coordinates

$$p_i = \frac{\partial L}{\partial \dot{q}_i} \ , \ P_i = \frac{\partial L'}{\partial \dot{Q}_i}$$

and you plug this into E-L and get the EoM. Easy.

The Hamiltonian, on the other hand, offers no clear connection between p and q. You have a lot more freedom in that q need note even be a spatial coordinate, nor p related to the velocity of anything. But, if you forego that freedom and q is a generalized spatial coordinate, then...

Given some generalized coordinates q_i,
the momenta $p_i = \frac{\partial L}{\partial \dot{q}_i}$ are those required for $H(p,q)$

Does this mean we need to construct $L(q,\dot{q})$ every time we want to change coordinates with H?

No! There are 3 other ways...

In each case we start with steps 1 and 2.

For the first path, we take step 3 and note that momenta are usually easy to guess (e.g. $\vec{p} = m\dot{\vec{r}}$).

Path 1: "guess and check"
Guess your momenta $P(p,q)$

$$\text{e.g. } p_\phi = mR^2\dot{\phi} = L_z = xp_y - yp_x$$

and check the Poisson Brackets (necessary and sufficient)

$$\begin{aligned} [Q_j, Q_k]_{pq} &= 0 \;\;, \quad [P_j, P_k]_{pq} = 0 \;\;, \quad [P_j, Q_k]_{pq} = \delta_{jk} \\ \text{e.g.} \quad [\phi, \phi] &= [\tan^{-1}\left(\frac{-x}{y}\right), \tan^{-1}\left(\frac{-x}{y}\right)] = 0 \\ [p_\phi, p_\phi] &= 0 \quad ([f,f] = 0 \text{ for any } f) \end{aligned}$$

$$\begin{aligned} [p_\phi, \phi] &= [xp_y - yp_x, \tan^{-1}\left(\frac{x}{y}\right)] \\ &= x\,[p_y, \tan^{-1}\left(\frac{-x}{y}\right)] - y\,[p_x, \tan^{-1}\left(\frac{-x}{y}\right)] \\ &= x\frac{\partial}{\partial y}\tan^{-1}\left(\frac{-x}{y}\right) - y\frac{\partial}{\partial x}\tan^{-1}\left(\frac{-x}{y}\right) \\ &= \frac{x^2}{R^2} + \frac{y^2}{R^2} = 1 \end{aligned}$$

Of course, we only have one generalized coordinate, ϕ, in this example. In general, you will have $\frac{3}{2}n\left(n-1\right)$ non-trivial PB to compute which give zero, and n of them which give 1, to perform this check. If $n>2$, you'll need a computer or a free weekend.

Result:

$$H=\frac{p_x^2+p_y^2}{2m}+mgy\Rightarrow H'=\frac{p_\phi^2}{2mR^2}-mgR\cos\phi$$

Paths 2 and 3 are similar and require some back story. Remember that curious fact about Lagrangians that adding the total time derivative of a function doesn't change the equation of motion? (LL eq. 2.8) I promised we would get back to that and here we are.

$$\begin{aligned}\text{Recall: } L' &= L+\frac{d}{dt}f\left(q,t\right)\Rightarrow \text{same EoM}\\ \text{and } L\left(q,\dot{q},t\right) &= L'\left(Q,\dot{Q},t\right)\Rightarrow \text{same E-L}\\ \text{and } L &= p\dot{q}-H\ \ ,\ \ L'=P\dot{Q}-H'\\ \Rightarrow p\dot{q}-H &= P\dot{Q}-H'+\frac{d}{dt}F\left(q,Q,p,P,t\right)\end{aligned}$$

If we limit F to be a function of one old variable (p or q) and one new variable (P or Q) it is called a "generating function". There are 4 ways we can do this, each with its own implications for the transformation (from p, q to P, Q) that results. The general rules are

$$\begin{aligned}\text{for } F_1\left(q,Q\right)\qquad p_i &= \frac{\partial F_1}{\partial q_i}\ \ ,\ \ P_i=-\frac{\partial F_1}{\partial Q_i}\\ \text{for } F_2\left(q,P\right)\qquad p_i &= \frac{\partial F_2}{\partial q_i}\ \ ,\ \ Q_i=\frac{\partial F_2}{\partial P_i}\end{aligned}$$

$$
\begin{aligned}
\text{for } F_3\left(p,Q\right) \qquad q_i &= -\frac{\partial F_3}{\partial p_i} \;\;,\;\; P_i = -\frac{\partial F_3}{\partial Q_i} \\
\text{for } F_4\left(p,P\right) \qquad q_i &= -\frac{\partial F_4}{\partial p_i} \;\;,\;\; Q_i = \frac{\partial F_4}{\partial P_i}
\end{aligned}
$$

So, if you want to make a coordinate transform with a Hamiltonian, you either do it through the Lagrangian, you guess and check with Poisson Brackets, or you find a generating function.

Let's do this for our pendulum example. Given a coordinate transform from old to new, we use F_2

Given $\vec{Q}(\vec{q})$, use $F_2(q,P) = \vec{Q}(\vec{q}) \cdot \vec{P} = \sum Q_i(\vec{q})\, P_i$

$$
\Rightarrow Q_i = \frac{\partial F_2}{\partial P_i} = Q_i(q) \;\;,\;\; p_i = \frac{\partial F_2}{\partial q_i}
$$

For pendulum

$$
Q(\vec{q}) \;\;\Rightarrow\;\; \phi(x,y) = \tan^{-1}\left(\frac{-x}{y}\right)
$$

$$
F_2 = \vec{Q}(\vec{q}) \cdot \vec{P} \;\;\Rightarrow\;\; F_2(x,y,p_\phi) = \tan^{-1}\left(\frac{-x}{y}\right) p_\phi
$$

This generating function is constructed to make $\frac{\partial F_2}{\partial P_i}$ trivially give us the desired Q_i (point transform). The second differential gives us the new momenta.

$$
\begin{aligned}
p_x &= \frac{\partial F_2}{\partial x} = p_\phi \frac{\partial}{\partial x} \tan^{-1}\left(\frac{-x}{y}\right) = p_\phi \left(\frac{-y}{R^2}\right) = p_\phi \frac{\cos\phi}{R} \\
p_y &= \frac{\partial F_2}{\partial y} = p_\phi \frac{\partial}{\partial y} \tan^{-1}\left(\frac{-x}{y}\right) = p_\phi \left(\frac{x}{R^2}\right) = p_\phi \frac{\sin\phi}{R}
\end{aligned}
$$

Since I only have one new momenta and two old, this is over constrained, and both give the same answer. The cartesian momenta are

$$\begin{aligned} p_x &= m\dot{x} = mR\cos\phi\dot{\phi} \Rightarrow p_\phi = mR^2\dot{\phi} \\ p_y &= m\dot{y} = mR\sin\phi\dot{\phi} \end{aligned}$$

where we inverted either expression to get p_ϕ. This matches our guess, so we have $H(p,q)$.

We can also use the F_3 generator function in a similar way. Again, we trivially recover our point transform, with the first differential equation,

Given $\vec{q}\left(\vec{Q}\right)$, use $F_3(\vec{p},\vec{Q}) = -\vec{q}\left(\vec{Q}\right)\cdot\vec{p}$

$$\Rightarrow q_i = -\frac{\partial F_3}{\partial p_i} = q\left(\vec{Q}\right) \quad , \quad P_i = -\frac{\partial F_3}{\partial Q_i}$$

and the second gives us the new momenta P.

For pendulum

$$\begin{aligned} \vec{q}(\vec{Q}) &\rightarrow x = R\sin\phi \ , \ y = -R\cos\phi \\ F_3(p_x,p_y,\phi) &= -R\sin\phi\, p_x + R\cos\phi\, p_y \end{aligned}$$

$$\begin{aligned} p_\phi &= -\frac{\partial F_3}{\partial\phi} = R\left(\cos\phi\, p_x + \sin\phi\, p_y\right) \\ &= R\left(\cos\phi\,(m\dot{x}) + \sin\phi\,(m\dot{y})\right) \\ &= mR\left(\cos\phi\left(R\cos\phi\,\dot{\phi}\right) + \sin\phi\left(R\sin\phi\,\dot{\phi}\right)\right) \\ &= mR^2\dot{\phi} \end{aligned}$$

For our example, in which the coordinate transform is most easily expressed as $\vec{q}\left(\vec{Q}\right)$, this path through F_3 is the most direct way to go from step 2 to step 5 without passing through L (at the price of needing to invert $P_i = f(\vec{p}, \vec{q})$).

Of course, we have explored only a very limited range of generator functions. These needn't result in point transforms; the Hamiltonian is not limited like the Lagrangian to point transforms $Q(q) \Rightarrow \dot{Q}(q, \dot{q})$. Rather, you can have $Q(p, q)$ and $P(p, q)$.

For instance, let's try this...

Transform H from ϕ, p_ϕ

$$\begin{aligned} H(p_\phi, \phi) &= \frac{p_\phi^2}{2mR^2} - mgR\cos\phi \\ &\simeq \frac{p_\phi^2}{2I} + \frac{k}{2}\phi^2 + \text{const} \quad \text{for} \quad \phi \ll 1 \\ \text{with} \qquad & I = mR^2\,,\ k = mgR = \frac{gI}{R} = I\omega^2. \end{aligned}$$

Dropping the constant gives us the Hamiltonian of a simple harmonic oscillator with frequency $\omega = \sqrt{\frac{g}{R}}$.

$$\begin{aligned} \text{try} \quad F_1(\phi, \theta) &= \frac{I\omega\phi^2}{2\tan\theta} \quad \text{with} \ \ \omega = \sqrt{\frac{g}{R}} \\ p_\phi &= \frac{\partial F_1}{\partial \phi} = \frac{I\omega\phi}{\tan\theta} \\ \Rightarrow \tan\theta &= \frac{I\omega\phi}{p_\phi} \\ p_\theta &= -\frac{\partial F_1}{\partial \theta} = \frac{I\omega\phi^2}{2}\frac{\partial}{\partial\theta}\frac{-1}{\tan\theta} = \frac{I\omega\phi^2}{2\sin^2\theta} \end{aligned}$$

Now we need to write $H(p_\theta, \theta)$ based on $H(p_\phi, \phi)$, just like we got $H(p_\phi, \phi)$ from H in cartesian coordinates.

Find $H(p_\theta, \theta)$

$$
\begin{aligned}
H(p_\theta, \theta) &= \frac{p_\phi^2}{2I} + \frac{I\omega^2}{2}\phi^2 = \frac{I\omega^2\phi^2}{2\tan^2\theta} + \frac{I\omega^2\phi^2}{2} \\
\text{use} \qquad & \frac{1}{\tan^2\theta} + 1 = \frac{1}{\sin^2\theta} \Rightarrow H = \omega p_\theta
\end{aligned}
$$

Now that is a simple Hamiltonian!

EoM for θ

$$
\begin{aligned}
\dot{\theta} &= \frac{\partial H}{\partial p_\theta} = \omega \ , \ \dot{p}_\theta = -\frac{\partial H}{\partial \theta} = 0 \\
\theta &= \omega t + \theta_0 \ , \ p_\theta = \text{const}
\end{aligned}
$$

What does this mean physically? Let's return to our angular coordinate ϕ;

$$
\begin{aligned}
\phi &= \sqrt{\frac{2p_\theta}{I\omega}} \sin(\omega t + \theta_0) \\
p_\phi &= \sqrt{2p_\theta I\omega} \cos(\omega t + \theta_0) \\
\text{SHO:} \qquad & E = \frac{1}{2} I\omega^2 A^2 \Rightarrow p_\theta = \frac{E}{\omega}
\end{aligned}
$$

so θ is the phase of the oscillator, and p_θ is related to the energy of the oscillation. (Note that $H = E$ as expected.)

So this generator function moved us into a "cordinate" system where our "momentum" was actually energy (a constant) and "position" was actually the phase of the harmonic oscillator solution!

This would not work with a Lagrangian!

8.223 Classical Mechanics II
January IAP 2017