Massachusetts Institute of Technology

8.223, Classical Mechanics II

Exercises 2

11. For a particle described by $L = \frac{1}{2}mv^2 - U$,

    a) Show that if $U = U(\vec{x}), E = T + U$ is constant.

    b) Show that if $U = U(\vec{x}, \dot{\vec{x}})$, then E is conserved, but it is not necessarily $E = T + U$.

12. For $L = T_a + T_b - u(\vec{x}_a - \vec{x}_b)$, show that $\vec{P} = \vec{p}_a + \vec{p}_b$ is conserved. Hint: try defining new coordinates:

$$\vec{X} = \vec{x}_a + \vec{x}_b$$
$$\vec{x} = \vec{x}_a - \vec{x}_b.$$

13. For a two body system with particle 1 of mass m_1 located at $\vec{x}_1$ and particle 2 of mass m_2 located at $\vec{x}_2$, find the center of mass $\vec{R}$ in terms of m_1, m_2, $\vec{x}_1$ and $\vec{x}_2$. (This is not a trick question.)

14. a) Show that the change in coordinates of a particle (fixed position) from a small rotation by ϕ around the $\hat{z}$ axis may be written as

$$d\vec{x} = \vec{x}' - \vec{x} = (\phi\hat{z}) \times \vec{x}.$$

    b) OPTIONAL: Show that the change in coordinates $d\vec{x}$ for a small rotation around the axis $\vec{b}$ by ϕ may be written $d\vec{x} = (\phi\hat{b}) \times \vec{x}$.

15. Moving pendulum: A plane pendulum consists of a bob of mass m suspended by a massless rigid rod of length l that is hinged to a sled of mass M. The sled slides without friction on a horizontal rail (see Fig. ??). Gravity acts on both masses.

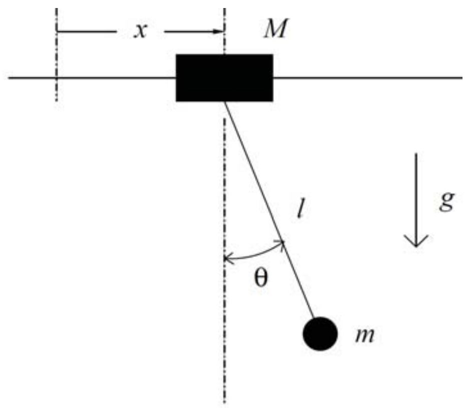

Figure 1: moving pendulum

a ×2) Write the Lagrangian for the system and derive the equations of motion.

b) At time $t = 0$ the bob and the sled, which had previously been at rest (with $\theta = 0$), are set in motion by a sharp tap delivered to the bob. The tap imparts a horizontal impulse $\Delta P = F\Delta t$ to the bob. Find expressions for the values of $\dot{\theta}$ and $\dot{x}$ just after the impulse. [Hint: consider both linear and angular momentum.]

16. ($\times 2$) Flyball Governor: In the system shown in Fig. 2, the particle m_2 moves on the vertical axis and the whole system rotates about this axis with a constant angular velocity Ω. Derive the Lagrangian of the system and obtain the equations of motion. Gravity pulls down on all masses.

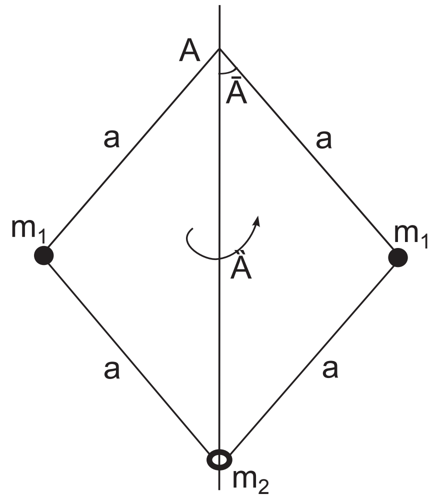

Figure 2: Centrifugal governor (aka, flyball governor)

17. ($\times 2$) Least time path: A particle can slide (without any friction) under gravity from point A$(0, 0, h)$ to point B$(X, 0, 0)$ along a curve $z(x)$ (Fig. 3). Find the path $z(x)$ that minimizes the time. The particle starts at rest from point A. [Hint: you can use conservation laws to simplify your calculation. You may encounter integrals of the form

$$\int \sqrt{\frac{a-t}{b+t}}\,\mathrm{d}t$$

which can be solved by substituting $t = a - (a + b)\sin^2\theta$.]

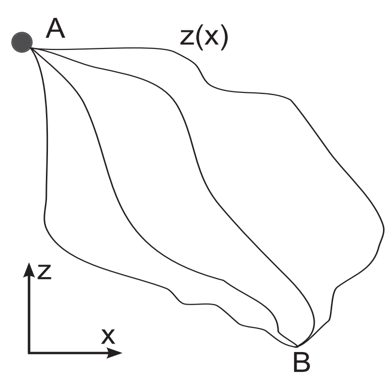

Figure 3: Least time path

18. Show that
$$\frac{d}{dt}(\vec{x}\cdot\dot{\vec{x}}) = v^2 + \vec{x}\cdot\ddot{\vec{x}}.$$
(also not a trick question)

19. Show that a potential $U = \kappa r^k$ results in a force $\vec{F} = -k\kappa r^{k-1}\hat{r}$.

20. Carry out the integration of Eq. 14.7 in Landau for $U = -\alpha/r$. Use the substitution $u = 1/r$ in Eq. 14.7 and show that $u = (\epsilon/p)\cos\phi + (1/p)$ solves the integral. Find p and ϵ in terms of M, μ, α and E. Hint: multiply the numerator and denominator by r^2.

21. Use $p/r = \epsilon\cos\phi + 1$ and your results from exercise 20 to show that $E > 0$ corresponds to a hyperbola, $E = 0$ corresponds to a parabola, and $E < 0$ corresponds to an ellipse.

22. There are plenty of asteroids flying around in the universe. For an asteroid coming from infinity with velocity v_∞, find the effective total cross-section for the asteroid to hit the earth (due to the gravitational attraction between them). You can assume that the asteroid is much smaller than the earth. What happens in the limit $v_\infty = \infty$? Interpret the result.

    Hint: Treat this as a scattering problem with the asteroids all moving in the same direction (as in a beam of particles), and compute the maximum impact factor for an asteroid to hit the Earth, b_{max}. The cross-section is then $\sigma = \pi b_{\text{max}}^2$.