Chapter 2

Rigid Body Dynamics

2.1 Coordinates of a Rigid Body

A set of N particles forms a *rigid body* if the distance between any 2 particles is fixed:

$$r_{ij} \equiv |\mathbf{r}_i - \mathbf{r}_j| = c_{ij} = \text{ constant.} \tag{2.1}$$

Given these constraints, how many generalized coordinates are there?

If we know 3 non-collinear points in the body, the remaing points are fully determined by triangulation. The first point has 3 coordinates for translation in 3 dimensions. The second point has 2 coordinates for spherical rotation about the first point, as r_{12} is fixed. The third point has one coordinate for circular rotation about the axis of $\mathbf{r}_{12}$, as r_{13} and r_{23} are fixed. Hence, there are *6 independent coordinates*, as represented in Fig. 2.1. This result is independent of N, so this also applies to a continuous body (in the limit of $N \to \infty$).

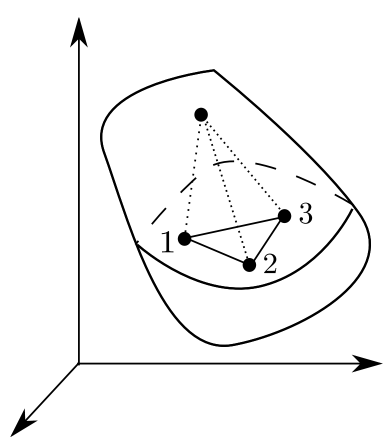


Figure 2.1: 3 non-collinear points can be fully determined by using only 6 coordinates. Since the distances between any two other points are fixed in the rigid body, any other point of the body is fully determined by the distance to these 3 points.

The translations of the body require three spatial coordinates. These translations can be taken from any fixed point in the body. Typically the fixed point is the center of mass (CM), defined as:

$$\mathbf{R} = \frac{1}{M} \sum_i m_i \mathbf{r}_i, \tag{2.2}$$

where m_i is the mass of the i-th particle and $\mathbf{r}_i$ the position of that particle with respect to a fixed origin and set of axes (which will notationally be unprimed) as in Fig. 2.2. In the case of a continuous body, this definition generalizes as:

$$\mathbf{R} = \frac{1}{M} \int_{\mathcal{V}} \mathbf{r} \rho(\mathbf{r}) \, d\mathcal{V}, \tag{2.3}$$

where $\rho(\boldsymbol{r})$ is the mass density at position $\boldsymbol{r}$ and we integrate over the volume $\mathcal{V}$.

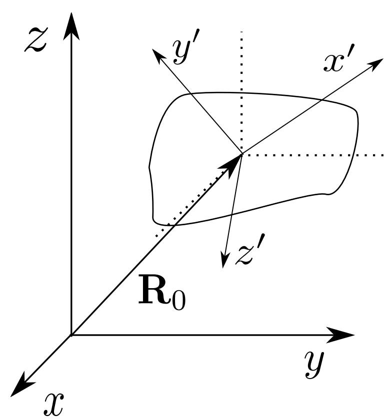


Figure 2.2: The three translational coordinates correspond to the position of the Center of Mass, and the three rotational coordinates correspond to the three angles necessary to define the orientation of the axis fixed with the body.

Rotations of the body are taken by fixing axes with respect to the body (we will denote these body fixed axes with primes) and describing their orientation with respect to the unprimed axes by 3 angles (ϕ, θ, ψ).

A particularly useful choice of angles are called *Euler angles*. The angle ϕ is taken as a rotation about the z-axis, forming new $\tilde{x}$- and $\tilde{y}$-axes while leaving the z-axis unchanged, as shown in Fig. 2.3. The angle θ is then taken as a rotation about the $\tilde{x}$-axis, forming new $\tilde{y}'$- and z'-axes while leaving the $\tilde{x}$-axis unchanged, as shown in Fig. 2.4. Finally, the angle ψ is taken as a rotation about the z'-axis, forming new x'- and y'-axes while leaving the z'-axis unchanged, as shown in Fig. 2.5. (The $\tilde{x}$-axis is called the line of nodes, as it is the intersection of the xy- and $\tilde{x}\tilde{y}$-planes.)

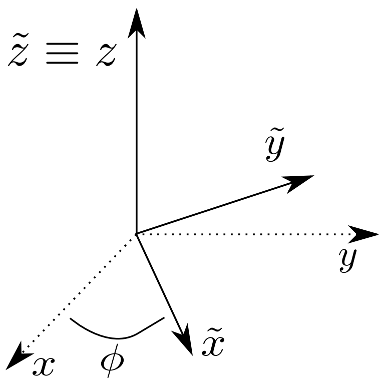


Figure 2.3: First rotation is by ϕ around the original z axis.

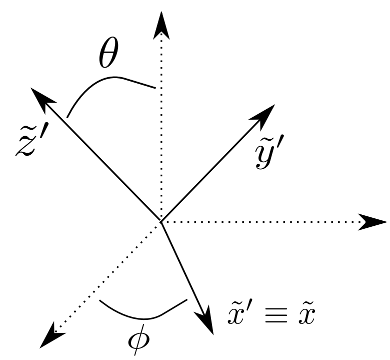


Figure 2.4: Second rotation is by θ around the intermediate $\tilde{x}$ axis.

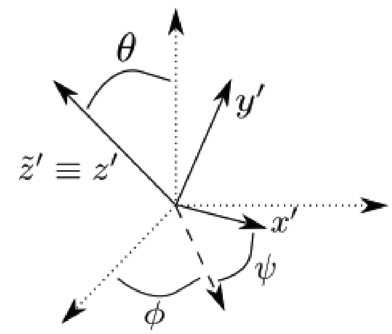


Figure 2.5: Final rotation is by ψ around the final z' axis.

Rotations can be described by 3×3 matrices U. This means each rotation step can be described as a matrix multiplication. Where $\mathbf{r} = (x, y, z)$, then

$$\tilde{\mathbf{r}} = U_\phi \mathbf{r} = \begin{bmatrix} \cos(\phi) & \sin(\phi) & 0 \\ -\sin(\phi) & \cos(\phi) & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix}. \tag{2.4}$$

Similar transformations can be written for the other terms:

$$\tilde{\mathbf{r}}' = U_\theta \tilde{\mathbf{r}} \qquad , \qquad \mathbf{r}' = U_\psi \tilde{\mathbf{r}}' = U_\psi U_\theta \tilde{\mathbf{r}} = U_\psi U_\theta U_\phi \mathbf{r}.$$

Defining the total transformation as U, it can be written as:

$$U \equiv U_\psi U_\theta U_\phi \quad \Rightarrow \quad \mathbf{r}' = U\mathbf{r}. \tag{2.5}$$

Care is required with the order of the terms since the matrices don't commute. Writing U out explicitly:

$$U = \begin{bmatrix} \cos(\psi) & \sin(\psi) & 0 \\ -\sin(\psi) & \cos(\psi) & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos(\theta) & \sin(\theta) \\ 0 & -\sin(\theta) & \cos(\theta) \end{bmatrix} \begin{bmatrix} \cos(\phi) & \sin(\phi) & 0 \\ -\sin(\phi) & \cos(\phi) & 0 \\ 0 & 0 & 1 \end{bmatrix}. \tag{2.6}$$

All rotation matrices, including U_ϕ, U_θ, U_ψ, and U are *orthogonal*. Orthogonal matrices W satisfy

$$W^\top W = W W^\top = \mathbb{1} \quad \Leftrightarrow \quad W^\top = W^{-1}, \tag{2.7}$$

where $\mathbb{1}$ refers to the identity matrix and $\top$ to the transpose. This ensures that the length of a vector is invariant under rotations:

$$\mathbf{r}'^2 = \mathbf{r}^\top (W^\top W) \mathbf{r} = \mathbf{r}^2. \tag{2.8}$$

Orthogonal matrices W have 9 entries but need to fulfill 6 conditions from orthogonality, leaving only 3 free parameters, corresponding to the 3 angles necessary to determine the rotation.

We can also view $\mathbf{r}' = U\mathbf{r}$ as a transformation from the vector $\mathbf{r}$ to the vector $\mathbf{r}'$ in the same coordinate system. This is an active transformation, as opposed to the previous perspective which was a passive transformation.

Finally, note that inversions like

$$U = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix} \qquad \text{or} \qquad U = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix} \tag{2.9}$$

are not rotations. These have $\det(U) = -1$, so they can be forbidden by demanding that $\det(U) = 1$. All orthogonal matrices have $\det(W) = \pm 1$ because $\det(W^\top W) = (\det(W))^2 = 1$. In the language of group theory, the restriction to $\det(W) = 1$ gives the special orthogonal group $SO(3)$ as opposed to simply $O(3)$, the orthogonal group. We disregard the $\det(U) = -1$ subset of transformations because it is impossible for the system to undergo these transformations continuously without the distance between the particles changing in the process, so it would no longer be a rigid body.

Intuitively, we could rotate the coordinates (x, y, z) directly into the coordinates (x', y', z') by picking the right axis of rotation. In fact, the *Euler theorem* states that a general displacement of a rigid body with one point fixed is a rotation about *some axis*. This theorem will be true if a general rotation U leaves some axis fixed, which is satisfied by

$$U\mathbf{r} = \mathbf{r} \tag{2.10}$$

for any point $\mathbf{r}$ on this axis. This is an eigenvalue equation for U with eigenvalue 1. To better understand this, we need to develop a little linear algebra.

Although the notion of an eigenvalue equation generally holds for linear operators, for now the discussion will be restricted to orthogonal rotation matrices U. The eigenvalue equation is

$$U\boldsymbol{\xi} = \lambda\boldsymbol{\xi}, \tag{2.11}$$

where $\boldsymbol{\xi}$ is an eigenvector and λ is the associated eigenvalue. Rewriting this as

$$(U - \lambda\mathbb{1})\boldsymbol{\xi} = 0 \tag{2.12}$$

requires that $\det(U - \lambda\mathbb{1}) = 0$, so that $U - \lambda\mathbb{1}$ is not invertible and the solution can be non-trivial, $\boldsymbol{\xi} \neq 0$. $\det(U - \lambda\mathbb{1}) = 0$ is a cubic equation in λ, which has 3 solutions, which are the eigenvalues λ_α for $\alpha \in \{1, 2, 3\}$. The associated eigenvectors are $\boldsymbol{\xi}^{(\alpha)}$ and satisfy

$$U\boldsymbol{\xi}^{(\alpha)} = \lambda_\alpha \boldsymbol{\xi}^{(\alpha)}, \tag{2.13}$$

where no implicit sum over repeated indices is taken. Forming a matrix from the resulting eigenvectors as columns:

$$X = \begin{bmatrix} \uparrow & \uparrow & \uparrow \\ \boldsymbol{\xi}^{(1)} & \boldsymbol{\xi}^{(2)} & \boldsymbol{\xi}^{(3)} \\ \downarrow & \downarrow & \downarrow \end{bmatrix} \tag{2.14}$$

then we can rewrite Eq.(2.13) as

$$UX = X \cdot \mathrm{diag}(\lambda_1, \lambda_2, \lambda_3) \Rightarrow X^{-1}UX = \mathrm{diag}(\lambda_1, \lambda_2, \lambda_3) \tag{2.15}$$

This means X diagonalizes U. Since U is orthogonal, the matrix X is unitary ($X^\dagger X = XX^\dagger = \mathbb{1}$). Note that $^\top$ indicates transposition whereas † indicates Hermitian conjugation (i.e. complex conjugation * combined with transposition $^\top$).

Next we note that since $\det(U) = 1$, then $\lambda_1\lambda_2\lambda_3 = 1$. Additionally, $|\lambda_\alpha|^2 = 1$ for each α because:

$$U\boldsymbol{\xi} = \lambda\boldsymbol{\xi} \quad \Rightarrow \quad \boldsymbol{\xi}^\dagger U^\top = \lambda^\star \boldsymbol{\xi}^\dagger \quad \Rightarrow \quad \boldsymbol{\xi}^\dagger\boldsymbol{\xi} = \boldsymbol{\xi}^\dagger U^\top U \boldsymbol{\xi} = |\lambda|^2 \boldsymbol{\xi}^\dagger \boldsymbol{\xi}\,. \tag{2.16}$$

Finally, if λ is an eigenvalue, then so is $\lambda^\star$:

$$U\boldsymbol{\xi} = \lambda\boldsymbol{\xi} \Rightarrow U\boldsymbol{\xi}^\star = \lambda^\star\boldsymbol{\xi}^\star \tag{2.17}$$

where $\boldsymbol{\xi}^\star$ is still a column vector but with its elements undergoing complex conjugation with respect to $\boldsymbol{\xi}$. Without loss of generality, let us say for a rotation matrix U that $\lambda_2 = \lambda_3^\star$. Then $1 = \lambda_1|\lambda_2|^2 = \lambda_1$, so one of the eigenvalues is 1, giving Eq.(2.10), and thus proving Euler's Theorem. The associated eigenvector $\boldsymbol{\xi}^{(1)}$ to the eigenvalue $\lambda_1 = 1$ is the rotation axis, and if $\lambda_2 = \lambda_3^\star = e^{i\Phi}$ then Φ is the rotation angle about that axis.

In fact, we can make good use of our analysis of Euler's theorem. Together the rotation axis and rotation angle can be used to define the instantaneous *angular velocity* $\boldsymbol{\omega}(t)$ such that:

$$|\boldsymbol{\omega}| = \dot{\Phi} \qquad \text{and} \qquad \boldsymbol{\omega} \parallel \boldsymbol{\xi}^{(1)}\,. \tag{2.18}$$

The angular velocity will play an important role in our discussion of time dependence with rotating coordinates in the next section. If we consider several consecutive displacements of the rigid body, then each can have its own axis $\boldsymbol{\xi}^{(1)}$ and its own $\dot{\Phi}$, so $\boldsymbol{\omega}$ changes at each instance of time, and hence $\boldsymbol{\omega} = \boldsymbol{\omega}(t)$ (for the entire rigid body).

2.2 Time Evolution with Rotating Coordinates

Lets use unprimed axes (x, y, z) for the fixed (inertial) axes, with fixed basis vectors $\mathbf{e}_i$. We will also use primed axes (x', y', z') for the body axes with basis vectors $\mathbf{e}'_i$.

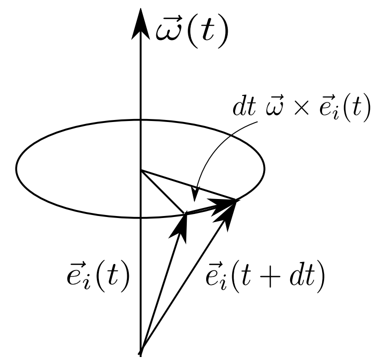
If we consider *any* vector then it can be decomposed with either set of basis vectors:

$$\mathbf{b} = \sum_i b_i \mathbf{e}_i = \sum_i b'_i \mathbf{e}'_i \,. \tag{2.19}$$

For fixed axes basis vectors by definition $\dot{\mathbf{e}}_i = 0$, while for those in the body frame,

$$\dot{\mathbf{e}}'_i = \boldsymbol{\omega}(t) \times \mathbf{e}'_i \tag{2.20}$$

meaning vectors of fixed length undergo a rotation at a time t. The derivation of this result is shown in the figure on the right, by considering the change to the vector after an infinitesimal time interval dt.



Summing over repeated indices, this means:

$$\dot{\mathbf{b}} = \dot{b}_i \mathbf{e}_i = \dot{b}'_i \mathbf{e}'_i + \boldsymbol{\omega}(t) \times (b'_i \mathbf{e}'_i) = \dot{b}'_i \mathbf{e}'_i + \boldsymbol{\omega}(t) \times \mathbf{b}$$

Defining $\dfrac{d}{dt}$ as the time evolution in the fixed (F) frame and $\dfrac{d_{\mathrm{R}}}{dt}$ the time evolution in the rotating/body (R) frame, then vectors evolve in time according to

$$\frac{d\mathbf{b}}{dt} = \frac{d_{\mathrm{R}}\mathbf{b}}{dt} + \boldsymbol{\omega} \times \mathbf{b}. \tag{2.21}$$

As a mnemonic we have the operation "$(d/dt) = d_R/dt + \boldsymbol{\omega}\times$" which can act on any vector.

Let us apply this to the position $\mathbf{r}$ of a particle of mass m, which gives

$$\frac{d\mathbf{r}}{dt} = \frac{d_{\mathrm{R}}\mathbf{r}}{dt} + \boldsymbol{\omega} \times \mathbf{r} \quad \Leftrightarrow \quad \mathbf{v}_{\mathrm{F}} = \mathbf{v}_{\mathrm{R}} + \boldsymbol{\omega} \times \mathbf{r}. \tag{2.22}$$

Taking another time derivative gives us the analog for acceleration:

$$\begin{aligned} \frac{\mathbf{F}}{m} = \frac{d\mathbf{v}_{\mathrm{F}}}{dt} &= \frac{d_{\mathrm{R}}\mathbf{v}_{\mathrm{F}}}{dt} + \boldsymbol{\omega} \times \mathbf{v}_{\mathrm{F}} \\ &= \frac{d_{\mathrm{R}}\mathbf{v}_{\mathrm{R}}}{dt} + \frac{d_{\mathrm{R}}\boldsymbol{\omega}}{dt} \times \mathbf{r} + \boldsymbol{\omega} \times \frac{d_{\mathrm{R}}\mathbf{r}}{dt} + \boldsymbol{\omega} \times \mathbf{v}_{\mathrm{R}} + \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r})\,. \end{aligned} \tag{2.23}$$

As $\frac{d_{\mathrm{R}}\mathbf{r}}{dt} = \mathbf{v}_{\mathrm{R}}$ is the velocity within the rotating body frame and $\frac{d_{\mathrm{R}}\mathbf{v}_{\mathrm{R}}}{dt} = \mathbf{a}_{\mathrm{R}}$ is the acceleration within the body frame, then

$$m\mathbf{a}_{\mathrm{R}} = \mathbf{F} - m\boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r}) - 2m\boldsymbol{\omega} \times \mathbf{v}_{\mathrm{R}} - m\frac{d_{\mathrm{R}}\boldsymbol{\omega}}{dt} \times \mathbf{r} \tag{2.24}$$

gives the acceleration in the body frame with respect to the forces that seem to be present in that frame. The terms $-m\boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r})$ and $-2m\boldsymbol{\omega} \times \mathbf{v}_{\mathrm{R}}$ are, respectively, the centrifugal and Coriolis ficticious forces respectively, while the last term $-m\frac{d_R\boldsymbol{\omega}}{dt} \times \mathbf{r}$ is a ficticious force that arises from non-uniform rotational motion, so that there is angular acceleration within the body frame. The same result could also have been obtained with the Euler-Lagrange equations for L in the rotating coordinates:

$$L = \frac{m}{2}(\dot{\boldsymbol{r}} + \boldsymbol{\omega} \times \boldsymbol{r})^2 - V\,, \tag{2.25}$$

and you will explore this on a problem set.

Note that the centrifugal term is radially outward and perpendicular to the rotation axis. To see this, decompose $\mathbf{r}$ into components parallel and perpendicular to ω, $\mathbf{r} = \mathbf{r}_\parallel + \mathbf{r}_\perp$, then $\boldsymbol{\omega} \times \boldsymbol{r} = \boldsymbol{\omega} \times \boldsymbol{r}_\perp$, so $-\boldsymbol{\omega} \times (\boldsymbol{\omega} \times \boldsymbol{r}_\perp) = w^2 \boldsymbol{r}_\perp$. This term is present for any rotating body. On the other hand, the Coriolis force is nonzero when there is a nonzero velocity in the rotating/body frame: $\mathbf{v}_R \neq 0$.

Example: Consider the impact of the Coriolis force on projectile motion on the rotating Earth, where the angular velocity is $\omega_{\rm Earth} = \frac{2\pi}{24\times 3600\,\rm s} \approx 7.3 \times 10^{-5}\,\rm s^{-1}$. We work out the cross-product $-\boldsymbol{\omega} \times \boldsymbol{v}_r$ as shown in Fig. 2.6 for a particle in the northern hemisphere, where ω points to the north pole. Thus a projectile in the northern/southern hemisphere would be perturbed to the right/left relative to its velocity direction $\boldsymbol{v}_r$.

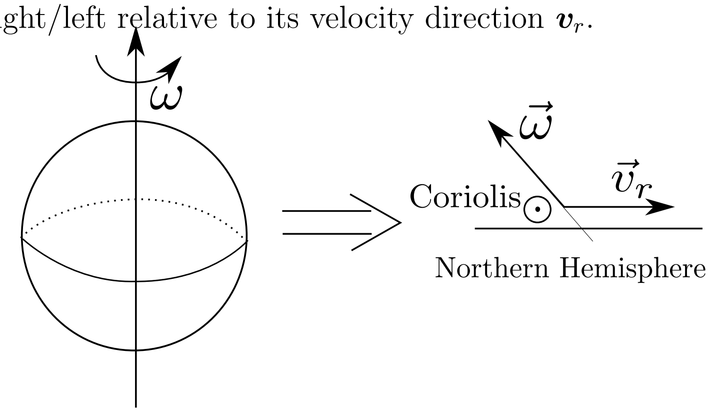
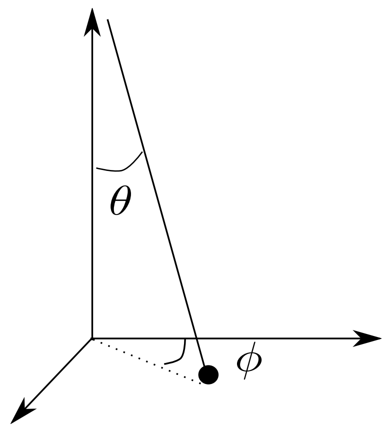
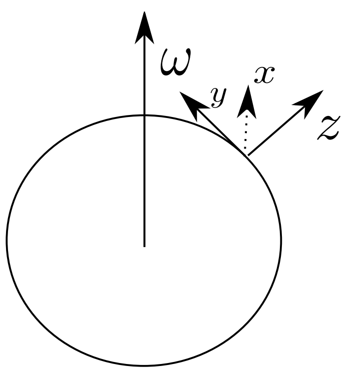


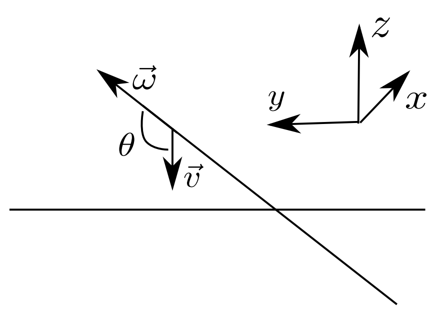
Figure 2.6: For a projectile, in the Northern Hemisphere, the Coriolis pushes it to its right, relative to its direction of motion.

Example: Consider a Foucault pendulum which hangs from a rigid rod, but is free to oscillate in two angular directions, as shown in Fig. 2.2. For $\theta \ll 1$ and working to first order in the small $\boldsymbol{\omega}$, the result derived from the Coriolis force gives $\dot{\phi} \approx \omega_{\rm Earth} \sin(\lambda)$. Here λ is the latitude angle measured from equator. The precession is clockwise in the northern hemisphere, and is maximized at the north pole where $\lambda = 90°$. (This is proven as a homework problem.)



Example: Consider the Coriolis deflection of a freely falling body on Earth in the northern hemisphere. We use the coordinate system shown below, where z is perpendicular to the surface of the earth and y is parallel to the earth's surface and points towards the north pole.





Working to first order in the small $\boldsymbol{\omega}$ gives us

$$m\mathbf{a}_{\mathrm{R}} = m\dot{\mathbf{v}}_{\mathrm{R}} = -mg\hat{z} - 2m\boldsymbol{\omega}\times\mathbf{v}, \tag{2.26}$$

where the centrifugal terms of order $O(\omega^2)$ are dropped. As an initial condition we take $\mathbf{v}(t=0) = v_0\hat{z}$. The term $-\boldsymbol{\omega}\times\mathbf{v}$ points along $\hat{x}$, so:

$$\ddot{z} = -g + O(\omega^2) \quad \Rightarrow \quad v_z = v_0 - gt \tag{2.27}$$

Moreover implementing the boundary condition that $\dot{x}(t=0) = 0$:

$$\ddot{x} = -2(\boldsymbol{\omega}\times\mathbf{v})_x = -2\omega\sin(\theta)v_z(t) \quad \Rightarrow \quad \dot{x} = -2\omega\sin(\theta)\left(v_0 t - \frac{g}{2}t^2\right). \tag{2.28}$$

Taking also $x(t=0) = 0$, and integrating one further time, the motion in the x direction is:

$$x(t) = -2\omega\sin(\theta)\left(\frac{v_0}{2}t^2 - \frac{g}{6}t^3\right). \tag{2.29}$$

Lets consider this effect for a couple simple cases. If the mass m is dropped from a height $z(t=0) = h_{\text{max}}$ with zero velocity, $v_0 = 0$, then:

$$z = h_{\text{max}} - \frac{g}{2}t^2 \tag{2.30}$$

and the mass reaches the floor at time

$$t_1 = \sqrt{\frac{2h_{\text{max}}}{g}}. \tag{2.31}$$

From Eq.(2.28) we see that $\dot{x} > 0$ for all t, and that:

$$x(t = t_1) = \frac{8\omega \sin(\theta) h_{\text{max}}^3}{3g^2} > 0.$$

However, if the mass m is thrown up with an initial $\dot{z}(t=0) = v_0 > 0$ from the ground $(z = 0)$, then :

$$z(t) = v_0 t - \frac{g}{2}t^2 > 0. \tag{2.32}$$

Here the particle rises to a maximum height $z = v_0^2/(2g)$ at time $t = v_0/g$, and then falls back to earth. Using Eq.(2.28) we see that $\dot{x} < 0$ for all t. If t_1 is the time it reaches the ground again ($t_1 = \frac{2v_0}{g}$), then:

$$x(t = t_1) = -\frac{4\omega \sin(\theta) v_0^3}{3g^2} < 0. \tag{2.33}$$

2.3 Kinetic Energy, Angular Momentum, and the Moment of Inertia Tensor for Rigid Bodies

Returning to rigid bodies, consider one built out of N fixed particles The kinetic energy is best expressed using CM coordinates, where $\mathbf{R}$ is the CM and we here take $\mathbf{r}_i$ to be the *displacement* of particle i relative to the CM. Once again making sums over repeated subscripts as implicit, the kinetic energy (T) of the system is given by:

$$T = \frac{1}{2}M\dot{\mathbf{R}}^2 + \frac{1}{2}m_i\dot{\mathbf{r}}_i^2. \tag{2.34}$$

As the body is rigid, then points cannot translate relative to the body but can only rotate so that $\dot{\mathbf{r}}_i = \boldsymbol{\omega} \times \mathbf{r}_i$. The rotational kinetic energy is then

$$T_{\text{R}} = \frac{1}{2}m_i\dot{\mathbf{r}}_i^2 = \frac{1}{2}m_i(\boldsymbol{\omega} \times \mathbf{r}_i)^2 = \frac{1}{2}m_i\big[\boldsymbol{\omega}^2\mathbf{r}_i^2 - (\boldsymbol{\omega} \cdot \mathbf{r}_i)^2\big]. \tag{2.35}$$

Labeling Cartesian indices with a and b to reserve i and j for particle indices, then we can write out this result making the indicies all explict as

$$T_{\mathrm{R}} = \frac{1}{2}\sum_{i,a,b} m_i(\delta_{ab}\mathbf{r}_i^2 - r_{ia}r_{ib})\omega_a\omega_b. \tag{2.36}$$

It is convenient to separate out the parts in this formula that depend on the shape and distributions of masses in the body by defining the *moment of inertia tensor* $\hat{I}$ for the discrete body as

$$\hat{I}_{ab} \equiv \sum_i m_i(\delta_{ab}\mathbf{r}_i^2 - r_{ia}r_{ib})\,. \tag{2.37}$$

The analog for a continuous body of mass density $\rho(\boldsymbol{r})$ is:

$$\hat{I}_{ab} \equiv \int_{\mathcal{V}} (\mathbf{r}^2\delta_{ab} - r_a r_b)\rho(\mathbf{r})\, d\mathcal{V}. \tag{2.38}$$

In terms of the moment of inertia tensor, the kinetic energy from rotation can now be written as:

$$T_{\mathrm{R}} = \frac{1}{2}\sum_{a,b} \hat{I}_{ab}\omega_a\omega_b = \frac{1}{2}\boldsymbol{\omega}\cdot\hat{I}\cdot\boldsymbol{\omega}\,, \tag{2.39}$$

where in the last step we adopt a convenient matrix multiplication notation.

The moment of inertia tensor can be written with its components as a matrix in the form

$$\hat{I} = \sum_i m_i \begin{bmatrix} y_i^2+z_i^2 & -x_iy_i & -x_iz_i \\ -x_iy_i & x_i^2+z_i^2 & -y_iz_i \\ -x_iz_i & -y_iz_i & x_i^2+y_i^2 \end{bmatrix}, \tag{2.40}$$

where the diagonal terms are the “moments of inertia” and the off-diagonal terms are the “products of inertia”. Note also that $\hat{I}$ is symmetric in any basis, so $\hat{I}_{ab} = \hat{I}_{ba}$.

Special case: if the rotation happens about only one axis which can be defined as the z-axis for convenience so that $\boldsymbol{\omega} = (0,0,\omega)$, then $T_{\mathrm{R}} = \frac{1}{2}\hat{I}_{zz}\omega^2$ which reproduces the simpler and more familiar scalar form of the moment of inertia.

Lets now let $\mathbf{r}_i$ be measured from a stationary point in the rigid body, which need not necessarily be the CM. The *angular momentum* can be calculated about this fixed point. Since $\mathbf{v}_i = \boldsymbol{\omega}\times\mathbf{r}_i$, we can write the angular momentum as:

$$\mathbf{L} = m_i\mathbf{r}_i\times\mathbf{v}_i = m_i\mathbf{r}_i\times(\boldsymbol{\omega}\times\mathbf{r}_i) = m_i\big[\mathbf{r}_i^2\boldsymbol{\omega} - (\boldsymbol{\omega}\cdot\mathbf{r}_i)\mathbf{r}_i\big]\,. \tag{2.41}$$

Writing out the components

$$L_a = \sum_i m_i(\mathbf{r}_i^2\omega_a - (\boldsymbol{\omega}\cdot\mathbf{r}_i)r_{ia}) = \sum_{i,b}\omega_b m_i(\delta_{ab}\mathbf{r}_i^2 - r_{ia}r_{ib}) = \sum_b \hat{I}_{ab}\omega_b, \tag{2.42}$$

which translates to the matrix equation:

$$\mathbf{L} = \hat{I} \cdot \boldsymbol{\omega} \,. \tag{2.43}$$

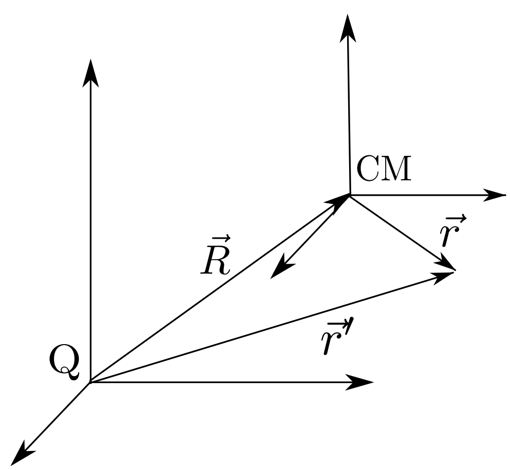
This allows us to write the corresponding rotational kinetic energy as:

$$T_{\mathrm{R}} = \frac{1}{2}\boldsymbol{\omega} \cdot \mathbf{L}. \tag{2.44}$$

Note that in general, $\mathbf{L}$ is *not* parallel to $\boldsymbol{\omega}$. We will see an explicit example of this below. Also note that the formula that we used for $\hat{I}$ in this case is the same as we gave above. We use these formulas whether or not $\mathbf{r}_i$ is taken with respect to the CM.

It is useful to pause to see what precisely the calculation of $\hat{I}$ depends on. Since it involves components of the vectors r_i it depends on *the choice of the origin* for the rotation. Furthermore the entries of the matrix $\hat{I}_{ab}$ obviously depend on the *orientation of the axes* used to define the components labeled by a and b. Given this, it is natural to ask whether given the result for $\hat{I}_{ab}$ with one choice of axes and orientation, whether we can determine an $\hat{I}'_{a'b'}$ for a different origin and axes orientation. This is always possible with the help of a couple of theorems.

The parallel axis theorem: Given $\hat{I}^{\mathrm{CM}}$ about the CM, it is simple to find $\hat{I}^{\mathrm{Q}}$ about a different point Q with the *same* orientation for the axes. Referring to the figure below,



we define $\mathbf{r}'_i$ as the coordinate of a particle i in the rigid body with respect to point Q and $\mathbf{r}_i$ to be the coordinate of that particle with respect to the CM, so that:

$$\mathbf{r}'_i = \mathbf{R} + \mathbf{r}_i \,. \tag{2.45}$$

By definition of the CM:

$$\sum_i m_i \mathbf{r}_i = 0 \qquad \text{and we let} \qquad M = \sum_i m_i. \tag{2.46}$$

The tensor of inertia around the new axis is then:

$$\hat{I}^{\mathrm{Q}}_{ab} = m_i(\delta_{ab}\mathbf{r}'^2_i - r'_{ia}r'_{ib}) \tag{2.47}$$
$$= m_i(\delta_{ab}(\mathbf{r}_i^2 + 2\mathbf{r}_i\cdot\mathbf{R} + \mathbf{R}^2 - \mathbf{r}_{ia}\mathbf{r}_{ib} - \mathbf{r}_{ia}\mathbf{R}_b - \mathbf{R}_a\mathbf{r}_{ib} - \mathbf{R}_a\mathbf{R}_b)\,, \tag{2.48}$$

where the cross terms involving a single $\boldsymbol{r}_i$ or single component $\boldsymbol{r}_{ia}$ sum up to zero by Eq.(2.46). The terms quadratic in $\boldsymbol{r}$ are recognized as giving the moment of inertia tensor about the CM. This gives the *parallel axis theorem* for translating the origin:

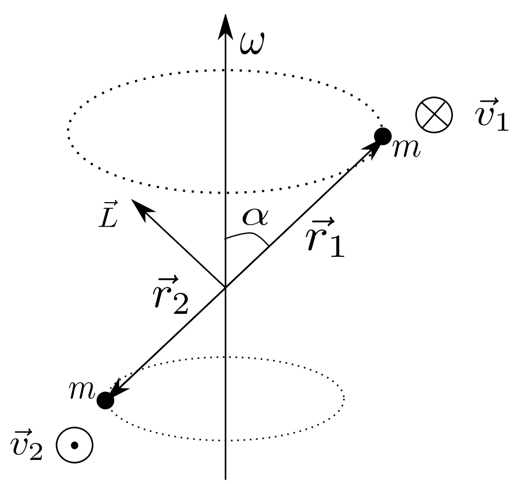
$$\hat{I}^{\mathrm{Q}}_{ab} = M(\delta_{ab}\mathbf{R}^2 - \boldsymbol{R}_a\boldsymbol{R}_b) + \hat{I}^{\mathrm{CM}}_{ab}, \tag{2.49}$$

If we wish to carry out a translation between P and Q, neither of which is the CM, then we can simply use this formula twice. Another formula can be obtained by projecting the parallel axis onto a specific axis $\hat{n}$ where $\hat{n}^2 = 1$ (giving a result that may be familiar from an earlier classical mechanics course):

$$\hat{n}\cdot\hat{I}^{\mathrm{Q}}\cdot\hat{n} = M(\mathbf{R}^2 - (\hat{n}\cdot\mathbf{R})^2) + \hat{n}\cdot\hat{I}^{\mathrm{CM}}\cdot\hat{n} = M\mathrm{R}^2[1-\cos^2(\theta)] + \hat{n}\cdot\hat{I}^{\mathrm{CM}}\cdot\hat{n}$$
$$= M\mathbf{R}^2\sin^2(\theta) + \hat{n}\cdot\hat{I}^{\mathrm{CM}}\cdot\hat{n} \tag{2.50}$$

where $\hat{n}\cdot\mathbf{R} \equiv R\cos(\theta)$.

Example: Lets consider an example of the calculation of $\hat{I}$ for a situation where $\boldsymbol{L}$ is not parallel to $\boldsymbol{\omega}$. Consider a dumbbell made of 2 identical point passes m attached by a massless rigid rod (but with different separations r_1 and r_2 from the axis of rotation), spinning so that $\boldsymbol{\omega} = \omega\hat{z}$ and so that the rod makes an angle α with the axis of rotation, as shown



We define body axes where the masses lie in the yz-plane. Here,

$$\mathbf{r}_1 = (0, r_1\sin\alpha, r_1\cos\alpha) \text{ and } \mathbf{r}_2 = (0, -r_2\sin\alpha, -r_2\cos\alpha). \tag{2.51}$$

Then using the definition of the moment inertia tensor:

$$
\begin{aligned}
I_{zz} &= m(x_1^2 + y_1^2) + m(x_2^2 + y_2^2) = m(r_1^2 + r_2^2)\sin^2\alpha \\
I_{xx} &= m(y_1^2 + z_1^2) + m(y_2^2 + z_2^2) = m(r_1^2 + r_2^2) \\
I_{yy} &= m(x_1^2 + z_1^2) + m(x_2^2 + z_2^2) = m(r_1^2 + r_2^2)\cos^2\alpha \\
I_{xy} &= I_{yx} = m(-x_1 y_1 - x_2 y_2) = 0 \\
I_{xz} &= I_{zx} = m(-x_1 z_1 - x_2 z_2) = 0 \\
I_{yz} &= I_{zy} = m(-y_1 z_1 - y_2 z_2) = -m(r_1^2 + r_2^2)\sin\alpha\cos\alpha
\end{aligned}
\tag{2.52}
$$

Plugging these into $\mathbf{L} = \hat{I}\cdot\boldsymbol{\omega}$, recalling that only ω_z is non-zero, this gives

$$
\boldsymbol{L} = (0, I_{yz}\omega, I_{zz}\omega). \tag{2.53}
$$

Thus in this example $\boldsymbol{L}$ is not parallel to $\boldsymbol{\omega}$.

Next, instead of translating the axes in a parallel manner, let us keep the origin fixed and rotate the axes according to an orthogonal rotation matrix U satisfying $U^\top U = UU^\top = 1$. Vectors are rotated as

$$
\mathbf{L}' = U\mathbf{L} \quad , \quad \boldsymbol{\omega}' = U\boldsymbol{\omega} \quad \text{and therefore} \quad \boldsymbol{\omega} = U^\top\boldsymbol{\omega}'. \tag{2.54}
$$

Putting these together

$$
\mathbf{L}' = U\hat{I}\cdot\boldsymbol{\omega} = (U\hat{I}U^\top)\cdot\boldsymbol{\omega}' \quad \Rightarrow \quad \hat{I}' = U\hat{I}U^\top, \tag{2.55}
$$

where $\hat{I}'$ is the new moment of inertia tensor. (The fact that it transforms this way *defines* it as a tensor.) This allows us to calculate the new moment of inertia tensor after a rotation.

For a real symmetric tensor $\hat{I}$, there always exists a rotation from an orthogonal matrix U that diagonalizes $\hat{I}$ giving a diagonal matrix $\hat{I}'$:

$$
\hat{I}_D = \begin{pmatrix} I_1 & 0 & 0 \\ 0 & I_2 & 0 \\ 0 & 0 & I_3 \end{pmatrix}. \tag{2.56}
$$

The entries of the diagonal moment of inertia tensor, I_α, are real and positive. This is just a special case of saying a Hermitian matrix can always be diagonalized by a unitary transformation (which is often derived in a Quantum Mechanics course as part of showing that a Hermitian matrix has real eigenvalues and orthogonal eigenvectors). The positivity of diagonal matrix follows immediately from the definition of the moment of inertia tensor for the situation with zero off-diagonal terms.

The axes that make $\hat{I}$ diagonal are called the *principal axes* and the components I_α are the *principal moments of inertia*. We find them by solving the eigenvalue problem

$$\hat{I} \cdot \boldsymbol{\xi} = \lambda\, \boldsymbol{\xi}, \tag{2.57}$$

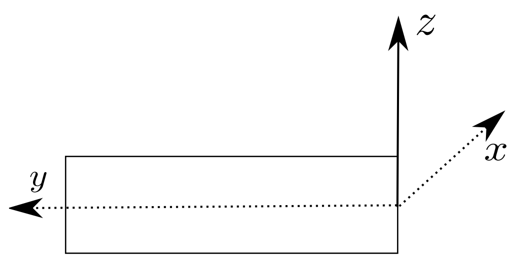
where the 3 eigenvalues λ give the principal moments of inertia I_α, and are obtained from solving $\det(\hat{I} - \lambda \mathbb{1}) = 0$. The corresponding 3 real *orthogonal* eigenvectors $\boldsymbol{\xi}^{(\alpha)}$ are the principal axes. Here $U^\top = \begin{bmatrix} \boldsymbol{\xi}^{(1)} & \boldsymbol{\xi}^{(2)} & \boldsymbol{\xi}^{(3)} \end{bmatrix}$, where the eigenvectors vectors fill out the columns. Then, without summing over repeated indices:

$$L_\alpha = I_\alpha \omega_\alpha \qquad \text{and} \qquad T = \frac{1}{2} \sum_\alpha I_\alpha\, \omega_\alpha^2\,, \tag{2.58}$$

where L_α and ω_α are the components of $\mathbf{L}$ and $\boldsymbol{\omega}$, respectively, evaluated along the principal axes.

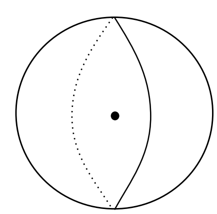
To summarize, for any choice of origin for any rigid body, there is a choice of axes that diagonalizes $\hat{I}$. For T to separate into translational and rotational parts, we must pick the origin to be the CM. Often, the principal axes can be identified by a symmetry of the body.

Example: for a thin rectangle lying in the yz-plane with one edge coinciding with the z-axis, and the origin chosen as shown below, then $I_{yz} = 0$ as the body is symmetric under $z \leftrightarrow -z$, while $I_{xz} = I_{xy} = 0$ as the body lies entirely within $x = 0$. Hence these are principal axes.

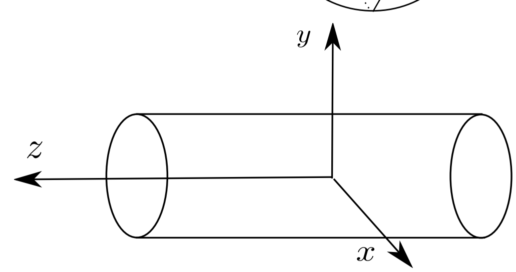


Sometimes, symmetry allows multiple choices for the principal axes.

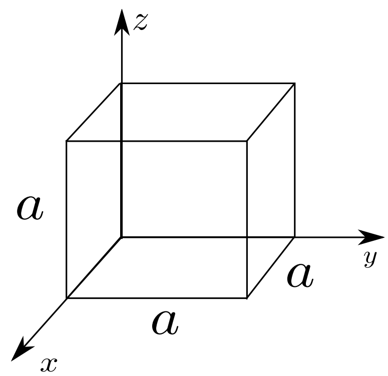
Example: for a sphere, any orthogonal axes through the origin are principal axes.



Example: for a cylinder whose central axis is aligned along the z-axis, because of rotational symmetry any choice of the x- and y-axes gives principal axes.



Example: Lets consider an example where the principal axes may not be apparent, which we can solve through the eigenvalue problem. Consider a uniform cube with sides of length a, mass m, and having the origin at one corner, as shown below.



By symmetry we have

$$I_{xx} = I_{yy} = I_{zz} = \frac{m}{a^3}\int_0^a\int_0^a\int_0^a (x^2+y^2)\,dx\,dy\,dz = \frac{2}{3}ma^2, \tag{2.59}$$
$$I_{xy} = I_{yz} = I_{xz} = \frac{m}{a^3}\int_0^a\int_0^a\int_0^a -xz\,dx\,dy\,dz = -\frac{1}{4}ma^2.$$

Thus the matrix is

$$\hat{I} = ma^2\begin{pmatrix} +\frac{2}{3} & -\frac{1}{4} & -\frac{1}{4} \\ -\frac{1}{4} & +\frac{2}{3} & -\frac{1}{4} \\ -\frac{1}{4} & -\frac{1}{4} & +\frac{2}{3} \end{pmatrix}. \tag{2.60}$$

The principal moments of inertia are found from

$$\det(\hat{I} - \lambda\mathbb{1}) = \left(\frac{11}{12}ma^2 - \lambda\right)^2\left(\frac{1}{6}ma^2 - \lambda\right) = 0\,. \tag{2.61}$$

This gives $I_1 = \lambda_1 = \frac{1}{6}ma^2$. Solving

$$(\hat{I} - \lambda_1\mathbb{1})\boldsymbol{\xi}^{(1)} = 0 \quad \text{we find} \quad \boldsymbol{\xi}^{(1)} = (1,1,1). \tag{2.62}$$

The remaining eigenvalues are degenerate:

$$I_2 = I_3 = \lambda_2 = \lambda_3 = \frac{1}{12}ma^2 \tag{2.63}$$

so there is some freedom in determining the corresponding principal axes from $(\hat{I} - \lambda_2\mathbb{1})\boldsymbol{\xi}^{(2,3)} = 0$, though they still should be orthogonal to each other (and $\boldsymbol{\xi}^{(1)}$). One example of a solution is:

$$\boldsymbol{\xi}^{(2)} = (1,-1,0) \quad \text{and} \quad \boldsymbol{\xi}^{(3)} = (1,1,-2) \tag{2.64}$$

Using these principal axes and the same origin, the moment of inertia tensor becomes

$$\hat{I}_D = \frac{ma^2}{6}\begin{pmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{2} & 0 \\ 0 & 0 & \frac{1}{2} \end{pmatrix} . \tag{2.65}$$

In contrast, if we had chosen the origin as the center of the cube, then one choice for the principal axes would have the same orientation, but with $\hat{I}_{\rm CM} = \frac{1}{6}ma^2\mathbb{1}$. This result could be obtained from Eq. (2.65) using the parallel axis theorem.

2.4 Euler Equations

Consider the rotational motion of a rigid body about a fixed point (which could be the CM but could also be another point). We aim to describe the motion of this rigid body by exploiting properties of the body frame. To simplify things as much as possible, for this fixed point, we choose the principal axes fixed in the body frame indexed by $\alpha \in \{1, 2, 3\}$. Using the relation between time derivatives in the inertial and rotating frames, the torque is then given by:

$$\boldsymbol{\tau} = \frac{d\mathbf{L}}{dt} = \frac{d_{\rm R}\mathbf{L}}{dt} + \boldsymbol{\omega} \times \mathbf{L} \tag{2.66}$$

where $\boldsymbol{\omega} = \boldsymbol{\omega}(t)$. For example:

$$\tau_1 = \frac{d_{\rm R}L_1}{dt} + \omega_2 L_3 - \omega_3 L_2. \tag{2.67}$$

Not summing over repeated indices and using the formula for angular momentum along the principal axes gives $L_\alpha = I_\alpha \omega_\alpha$. Since we have fixed moments of inertia within the body we have $d_R I_\alpha/dt = 0$. Note that $d\boldsymbol{\omega}/dt = d_R\boldsymbol{\omega}/dt + \boldsymbol{\omega} \times \boldsymbol{\omega} = d_R\boldsymbol{\omega}/dt$, so its rotating and inertial time derivatives are the same, and we can write $\dot{\omega}_\alpha$ without possible cause of confusion. Thus $d_R L_\alpha/dt = I_\alpha\dot{\omega}_\alpha$. This yields *the Euler equations*:

$$\begin{aligned} I_1\dot{\omega}_1 - (I_2 - I_3)\omega_2\omega_3 &= \tau_1 \\ I_2\dot{\omega}_2 - (I_3 - I_1)\omega_3\omega_1 &= \tau_2 \\ I_3\dot{\omega}_3 - (I_1 - I_2)\omega_1\omega_2 &= \tau_3 \end{aligned} \tag{2.68}$$

where in all of these $\boldsymbol{\omega}$ and $\boldsymbol{\tau}$ are calculated in the rotating/body frame. This can also be written as

$$\tau_\alpha = I_\alpha\,\dot{\omega}_\alpha + \epsilon_{\alpha lk}\,\omega_l\,\omega_k\,I_k\,, \tag{2.69}$$

with α fixed but a sum implied over the repeated l and k indicies. Here ϵ_{abc} is the fully antisymmetric Levi-Civita symbol.

Solving these equations gives $\omega_\alpha(t)$. Since the result is expressed in the body frame, rather than the inertial frame of the observer, this solution for $\boldsymbol{\omega}(t)$ may not always make the physical motion transparent. To fix this we can connect our solution to the Euler angles using the relations

$$\begin{aligned}\omega_1 &= \omega_{x'} = \dot{\phi}\,\sin\theta\,\sin\psi + \dot{\theta}\,\cos\psi\,,\\ \omega_2 &= \omega_{y'} = \dot{\phi}\,\sin\theta\,\cos\psi - \dot{\theta}\,\sin\psi\,,\\ \omega_3 &= \omega_{z'} = \dot{\phi}\,\cos\theta + \dot{\psi}.\end{aligned} \tag{2.70}$$

These results should be derived as exercise for the student.

Example: let us consider the stability of rigid-body free rotations ($\boldsymbol{\tau} = 0$). Is a rotation $\boldsymbol{\omega} = \omega_1 \mathbf{e}_1$ about the principal axis $\mathbf{e}_1$ stable?
Perturbations can be expressed by taking $\boldsymbol{\omega} = \omega_1\mathbf{e}_1 + \kappa_2\mathbf{e}_2 + \kappa_3\mathbf{e}_3$, where κ_2 and κ_3 are small and can be treated to 1^{st} order. The Euler equations are:

$$\dot{\omega}_1 = \frac{(I_2 - I_3)}{I_1}\,\kappa_2\kappa_3 = O(\kappa^2) \approx 0, \tag{2.71}$$

so ω_1 is constant at this order, and

$$\dot{\kappa}_2 = \frac{(I_3 - I_1)}{I_2}\,\omega_1\,\kappa_3 \qquad \text{and} \qquad \dot{\kappa}_3 = \frac{(I_1 - I_2)}{I_3}\,\omega_1\,\kappa_2\,. \tag{2.72}$$

Combining these two equations yields

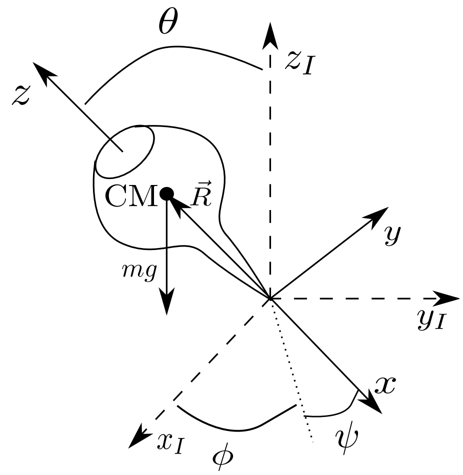
$$\ddot{\kappa}_2 = \left[\frac{(I_3 - I_1)(I_1 - I_2)\omega_1^2}{I_2 I_3}\right]\kappa_2. \tag{2.73}$$

The terms in the square bracket are all constant, and is either negative $= -w^2$ with an oscillating solution $\kappa_2 \propto \cos(wt + \phi)$, or is positive $= \alpha^2$ with exponential solutions $\kappa_2 \propto ae^{\alpha t} + be^{-\alpha t}$. If $I_1 < I_{2,3}$ or $I_{2,3} < I_1$ then the constant prefactor is negative, yielding stable oscillatory solutions. If instead $I_2 < I_1 < I_3$ or $I_3 < I_1 < I_2$ then the constant prefactor is positive, yielding an unstable exponentially growing component to their solution! This behavior can be demonstrated by spinning almost any object that has three distinct principal moments of inertia.

2.5 Symmetric Top with One Point Fixed

This section is devoted to a detailed analysis of a particular example that appears in many situations, the symmetric top with one point fixed, acted upon by a constant force.

Labeling the body axes as (x, y, z) and the fixed axes as $(x_\text{I}, y_\text{I}, z_\text{I})$, as depicted in the right, symmetry implies that $I_1 = I_2$, and we will assume that $I_{1,2} \neq I_3$. The Euler angles are as usual (ϕ, θ, ψ). From the figure we see that $\dot{\psi}$ is the rotation rate of the top about the (body) z-axis, $\dot{\phi}$ is the precession rate about the z_I fixed inertial axis, and $\dot{\theta}$ is the nutation rate by which the top may move away or towards the z_I axis. The Euler equations in this case are



$$
\begin{aligned}
I_1\dot{\omega}_1 - (I_2 - I_3)\omega_2\omega_3 &= \tau_1, \\
I_1\dot{\omega}_2 - (I_3 - I_1)\omega_3\omega_1 &= \tau_2, \\
I_3\dot{\omega}_3 = 0 &= \tau_3 \,.
\end{aligned}
\tag{2.74}
$$

Since the CM coordinate $\mathbf{R}$ is aligned along the z-axis there is no torque along z, $\tau_3 = 0$, leading to a constant ω_3.

There are two main cases that we will consider.

Case: $\boldsymbol{\tau} = 0$ and $\dot{\theta} = 0$

The first case we will consider is when $\boldsymbol{\tau} = 0$ (so there is no gravity) and $\dot{\theta} = 0$ (so there is no nutation). Then

$$\frac{d\mathbf{L}}{dt} = \boldsymbol{\tau} = 0 \quad \Rightarrow \quad \mathbf{L} = \text{ constant} \tag{2.75}$$

Let us define the constant:

$$\Omega \equiv \frac{I_3 - I_1}{I_1}\omega_3 \,. \tag{2.76}$$

Then the Euler equations for this situation reduce to:

$$\dot{\omega}_1 + \Omega\omega_2 = 0 \quad \text{and} \quad \dot{\omega}_2 - \Omega\omega_1 = 0 \,. \tag{2.77}$$

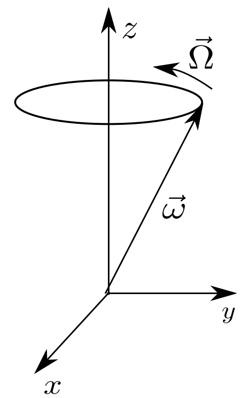
The simplest solution correspond to $\omega_1(t) = \omega_2(t) = 0$, where we just have a rotation about the z-axis. Here:

$$
\begin{aligned}
&\mathbf{L} = L_3\,\mathbf{e}_3 \quad \text{where} \quad L_3 = I_3\omega_3 \\
&\omega_1 = \omega_2 = 0 \quad \Rightarrow \quad \dot{\theta} = \dot{\phi} = 0 \text{ and } \dot{\psi} = \omega_3 \,.
\end{aligned}
\tag{2.78}
$$

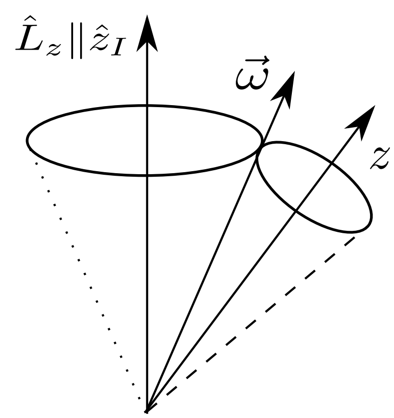
In this case $\mathbf{L} \parallel \boldsymbol{\omega}$. A more general situation is when $\mathbf{L}$ and $\boldsymbol{\omega}$ are not necessarily parallel, and ω_1 and ω_2 do not vanish. In this case Eq. (2.77) is solved by:

$$\omega_1 = C\sin(\Omega t + D) \quad \text{and} \quad \omega_2 = -C\cos(\Omega t + D). \tag{2.79}$$

The simple case corresponds to $C = 0$, so now we take $C > 0$ (since a sign can be accounted for by the constant phase D). This solution means $\boldsymbol{\omega}$ precesses about the body z-axis at the rate Ω, as pictured on the right. Since $\omega_1^2 + \omega_2^2$ is constant, the full $\omega = |\boldsymbol{\omega}|$ is constant, and is given by $\omega^2 = C^2 + \omega_3^2$.
The energy here is just rotational kinetic energy $T_{\rm R} = \frac{1}{2}\boldsymbol{\omega}\cdot\mathbf{L}$ which is constant too, since both $\boldsymbol{\omega}$ and $\boldsymbol{L}$ are constant. Thus $\boldsymbol{\omega}$ also precesses about $\mathbf{L}$.



We can picture this motion by thinking about a body cone that rolls around a cone in the fixed coordinate system, where in the case pictured with a larger cone about $\boldsymbol{L}$ we have $I_1 = I_2 > I_3$.



To obtain more explicit results for the motion we can relate Eq.(2.79) to Euler angles. Since $\dot\theta = 0$, we take $\theta = \theta_0$ to be constant. The other Euler angles come from:

$$\boldsymbol{\omega} = \begin{bmatrix} C\sin(\Omega t + D) \\ -C\cos(\Omega t + D) \\ \omega_3 \end{bmatrix} = \begin{bmatrix} \sin(\theta_0)\sin(\psi)\dot\phi \\ \sin(\theta_0)\cos(\psi)\dot\phi \\ \cos(\theta_0)\dot\phi + \dot\psi \end{bmatrix} . \tag{2.80}$$

Adding the squares of the 1st and 2nd components gives

$$C^2 = \sin^2(\theta_0)\ \dot\phi^2. \tag{2.81}$$

To be definite, take the positive square root of this equation to give

$$\dot\phi = \frac{C}{\sin(\theta_0)} \quad \Rightarrow \quad \phi = \frac{C}{\sin(\theta_0)}\, t + \phi_0. \tag{2.82}$$

The first two equations in Eq. (2.80) are then fully solved by taking $\psi = \pi - \Omega t - D$, so we find that both ϕ and ψ have linear dependence on time. Finally the third equation gives a relation between various constants

$$\omega_3 = C\cot(\theta_0) - \Omega. \tag{2.83}$$

Thus, we see that the solution has $\dot\phi$ and $\dot\psi$ are constants with $\dot\theta = 0$. If we had picked the opposite sign when solving Eq. (2.81) then we would have found similar results:

$$\dot\phi = -\frac{C}{\sin(\theta_0)} \quad \Rightarrow \quad \psi = -\Omega t - D \quad \text{and} \quad \omega_3 = -C\cot(\theta_0) - \Omega\,. \tag{2.84}$$

Case: $\boldsymbol{\tau} \neq 0$ and $\dot\theta \neq 0$

Now we consider the general case where $\boldsymbol{\tau} \neq 0$ and $\dot\theta \neq 0$. It is now more convenient to use the Lagrangian than the Euler equations directly. Since $I_1 = I_2$, using

$$T = \frac{1}{2}\left(I_1(\omega_1^2+\omega_2^2) + I_3\omega_3^2\right) \quad \text{and} \quad \boldsymbol{\omega} = \begin{bmatrix} \sin(\theta)\sin(\psi)\dot\phi + \cos(\psi)\dot\theta \\ \sin(\theta)\cos(\psi)\dot\phi - \sin(\psi)\dot\theta \\ \cos(\theta)\dot\phi + \dot\psi \end{bmatrix}, \tag{2.85}$$

gives us the kinetic energy

$$T = \frac{I_1}{2}\left(\dot\theta^2 + \sin^2\theta\,\dot\phi^2\right) + \frac{I_3}{2}\left(\dot\psi + \cos\theta\,\dot\phi\right)^2. \tag{2.86}$$

Moreover, $V = mgR\cos(\theta)$, so in the Lagrangian $L = T - V$ the variables ϕ and ψ are cyclic. This means that the momenta

$$p_\phi = \frac{\partial L}{\partial\dot\phi} = \left[I_1\sin^2(\theta) + I_3\cos^2(\theta)\right]\dot\phi + I_3\cos(\theta)\dot\psi \tag{2.87}$$

$$p_\psi = \frac{\partial L}{\partial\dot\psi} = I_3(\dot\psi + \cos(\theta)\dot\phi) = I_3\omega_3 \tag{2.88}$$

are conserved (constant). Here p_ψ is same as the angular momentum L_3 discussed in the case above. The torque is along the line of nodes, and p_ϕ and p_ψ correspond to two projections of $\mathbf{L}$ that are perpendicular to this torque (i.e. along $\hat z_{\mathrm{I}}$ and $\hat z$). Additionally, the energy is given by

$$E = T + V = \frac{I_1}{2}\left(\dot\theta^2 + \sin^2(\theta)\dot\phi^2\right) + \frac{I_3}{2}\left(\dot\psi + \cos(\theta)\dot\phi\right)^2 + mgR\cos(\theta) \tag{2.89}$$

and is also conserved. Solving the momentum equations, Eq. (2.87), for $\dot\phi$ and $\dot\psi$ gives

$$\begin{aligned} \dot\phi &= \frac{p_\phi - p_\psi\cos(\theta)}{I_1\sin^2(\theta)} \\ \dot\psi &= \frac{p_\psi}{I_3} - \frac{(p_\phi - p_\psi\cos(\theta))\cos(\theta)}{I_1\sin^2(\theta)}\,. \end{aligned} \tag{2.90}$$

Note that once we have a solution for $\theta(t)$ that these two equations then allow us to immediately obtain solutions for $\phi(t)$ and $\psi(t)$ by integration. Eq. (2.90) can be plugged into the energy formula to give

$$E = \frac{I_1}{2}\dot{\theta}^2 + \frac{(p_\phi - p_\psi \cos(\theta))^2}{2I_1 \sin^2(\theta)} + \frac{p_\psi^2}{2I_3} + mgR\cos(\theta), \tag{2.91}$$

which is a (nonlinear) differential equation for θ, since all other quantities that appear are simply constants. To simplify this result take $u = \cos(\theta)$ so that:

$$1 - u^2 = \sin^2(\theta)\,, \qquad \dot{u} = -\sin(\theta)\dot{\theta}\,, \qquad \dot{\theta}^2 = \frac{\dot{u}^2}{1-u^2}\,. \tag{2.92}$$

Putting all this together gives:

$$\frac{\dot{u}^2}{2} = \left(\frac{2EI_3 - p_\psi^2}{2I_1 I_3} - \frac{mgR}{I_1}u\right)\left(1-u^2\right) - \frac{1}{2}\left(\frac{p_\phi - p_\psi u}{I_1}\right)^2 \equiv -V_{\text{eff}}(u)\,, \tag{2.93}$$

which is a cubic polynomial that we've defined to be the effective potential $V_{\text{eff}}(u)$. The solution to this from

$$dt = \pm\frac{du}{\sqrt{-2V_{\text{eff}}(u)}} \tag{2.94}$$

yields a complicated elliptic function, from which it is hard to get intuition for the motion.

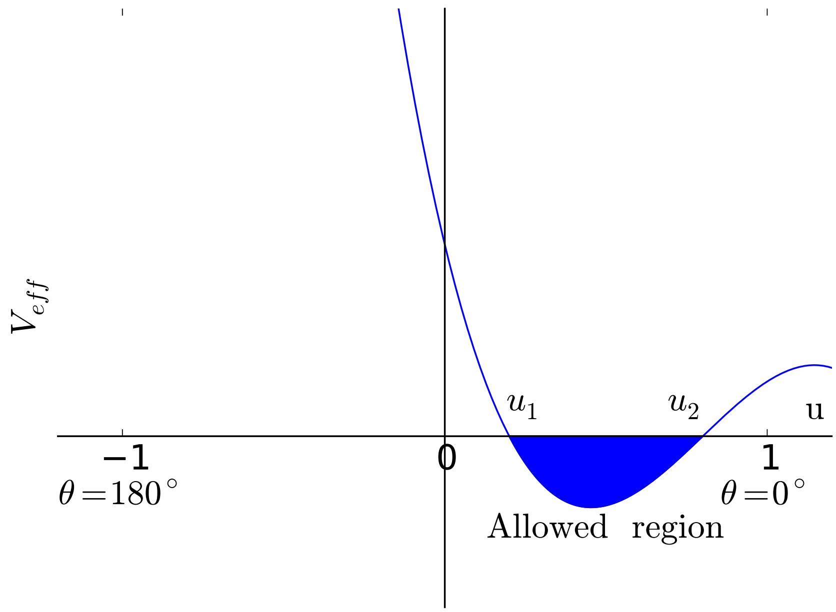


Figure 2.7: Allowed region for solutions for the top's nutation angle θ that solve Eq. (2.95).

Instead, we can look at the form of $V_{\text{eff}}(u)$, because

$$\frac{1}{2}\dot{u}^2 + V_{\text{eff}}(u) = 0 \tag{2.95}$$

is the equation for the energy of a particle of unit mass $m = 1$, kinetic energy $\dot{u}^2/2$, a potential $V_{\text{eff}}(u)$, and with vanishing total energy. The cubic equation will have in general three roots where $V_{\text{eff}}(u) = 0$. Since the kinetic energy is always positive or zero, the potential energy must be negative or zero in the physical region, and hence the particle can not pass through any of the roots. The roots therefore serve as turning points. Furthermore, physical solutions are bounded by $-1 \leq (u = \cos\theta) \leq 1$. While the precise values for the roots will depend on the initial conditions or values of E, p_ψ, and p_ϕ, we can still describe the solutions in a fairly generic manner.

Consider two roots u_1 and u_2 (corresponding respectively to some angles θ_1 and θ_2 as $u = \cos(\theta)$) satisfying $V_{\text{eff}}(u_1) = V_{\text{eff}}(u_2) = 0$, where $V_{\text{eff}}(u) < 0$ for $u_1 < u < u_2$; as shown in Fig. 2.7. We see that u_1 and u_2 correspond to the turning points of the motion. The region $u_1 < u < u_2$ corresponds to the region where the motion of our top lives and gives rise to a periodic nutation, where the solution bounces between the two turning points. Depending on the precise value of the various constants that appear in this V_{eff} this gives rise to different qualitative motions, with examples shown in Figs. 2.8–2.11. Recalling that $\dot{\phi} = (p_\phi - p_\psi u)/[I_1(1-u^2)]$, we see that the possible signs for $\dot{\phi}$ will depend on p_ϕ and p_ψ. In Fig. 2.8 the top nutates between θ_1 and θ_2 while always precessing in the same direction

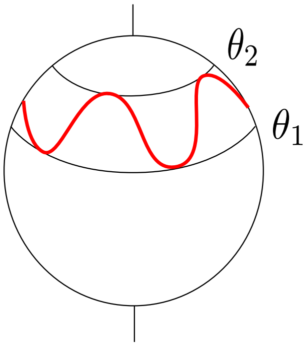


Figure 2.8: $\dot{\phi} > 0$

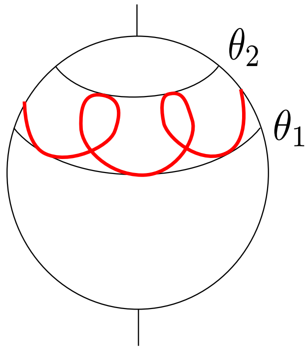


Figure 2.9: $\dot{\phi}$ has both signs

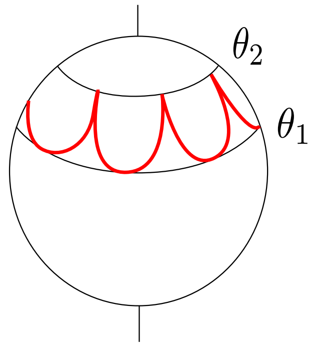


Figure 2.10: at θ_2 we have $\dot{\phi} = 0$, $\dot{\theta} = 0$

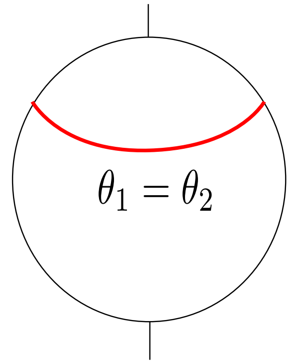


Figure 2.11: No nutation

with $\dot{\phi} > 0$, whereas in Fig. 2.9 the precession is also in the backward direction, $\dot{\phi} < 0$, for part of the range of motion. In Fig. 2.10 the top has $\dot{\phi} = 0$ at θ_2, before falling back down in the potential and gaining $\dot{\phi} > 0$ again. This figure also captures the case where we let go of a top at $\theta = \theta_2 \geq 0$ that initially has $\dot{\psi} > 0$ but $\dot{\phi} = 0$. Finally in Fig. 2.11 we have the situation where there is no nutation oscillation because the two angles coincide, $\theta_1 = \theta_2$.