Classical Mechanics III (8.09) Fall 2014
Assignment 3

Massachusetts Institute of Technology
Physics Department
September 22, 2014

Due September 29, 2014
6:00pm

Announcements

This week we continue our discussion of Rigid Bodies, including the general example of a heavy symmetric top precessing and nutating.

Reading Assignment for this week

- The main readings for Rigid Bodies are **Goldstein** Ch.4, sections 4.1, 4.2, 4.4, 4.6, 4.9. (Note that 4.3 is a linear algebra review.) The Ch.4 reading is on rigid body kinematics where many physics topics will be familiar to you. Goldstein emphasizes vector notation and discusses rotations as matrices. Next read Ch.5, sections 5.1, 5.3–5.7. Ch.5 is on rigid body dynamics. We will not cover Poinsot's construction, so you may skip this material on pages 201-205 of section 5.6. (Other sections from Ch.4 and Ch.5 may be of interest, but the above are the most important ones.)

- If you find the reading in **Goldstein** too dense, you should consider reading **Thornton & Marion** Ch.11. I particularly recommend Ch.11 section 10 on the force-free motion of a symmetric top.

- Our next item to discuss will be principal axes for oscillating motion. If you would like to read ahead, the reading for this will be **Goldstein** Ch.6 sections 6.1-6.4.

Problem Set 3

In these five problems you will study the dynamics of rigid bodies and the use of rotating coordinate systems. If you can obtain a result using symmetry you should!

1. **Rotation Angle in the Euler Theorem** [10 points]

In lecture we demonstrated that a general rotation can be thought of as a simple rotation about some fixed axis. In this problem you will fill in the last step of this analysis, namely showing that the angle Φ that appeared in those calculations is a rotation angle.

Lets first recall the setup. Consider a general rotation U so that $\vec{r}\,' = U\vec{r}$ with U an orthogonal matrix. Let $\vec{\xi}_i$ be the eigenvectors satisfying $U\vec{\xi}_i = \lambda_i\,\vec{\xi}_i$ where

$$\lambda_1 = e^{i\Phi}\,, \qquad \lambda_2 = e^{-i\Phi}\,, \qquad \lambda_3 = 1\,, \qquad \vec{\xi}_i^{\,\dagger}\cdot\vec{\xi}_j = \delta_{ij}\,.$$

Here $\Phi \neq 0$, and $\dagger$ means the complex conjugate and transpose. Also $\vec{\xi}_3$ is real, $\vec{\xi}_3^{\,*} = \vec{\xi}_3$, whereas the other two eigenvectors are complex satisfying $\vec{\xi}_1^{\,*} = \vec{\xi}_2$. Finally, the matrix $X = (\vec{\xi}_1\,\vec{\xi}_2\,\vec{\xi}_3)$ is unitary, $X^\dagger = X^{-1}$ and

$$X^\dagger U X = \begin{pmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{pmatrix}.$$

In order to demonstrate that U describes a simple rotation with rotation angle Φ we will find a set of coordinates that make this obvious. Since $U\vec{\xi}_3 = \vec{\xi}_3$, the rotation is about the axis given by the real vector $\vec{\xi}_3$, so we will pick $\vec{\xi}_3$ as our new z axis.

(a) [4 points] A natural choice for the other two axes might be $\vec{\xi}_1$ and $\vec{\xi}_2$. In this case the transformation to the new coordinates would be $\vec{r} = X\vec{s}$, where in the new coordinate system $\vec{\xi}_1 = \left(\begin{smallmatrix}1\\0\\0\end{smallmatrix}\right)$, $\vec{\xi}_2 = \left(\begin{smallmatrix}0\\1\\0\end{smallmatrix}\right)$, $\vec{\xi}_3 = \left(\begin{smallmatrix}0\\0\\1\end{smallmatrix}\right)$. BUT in our original coordinates $\vec{\xi}_1$ and $\vec{\xi}_2$ are complex so they are NOT valid choices for the axes. Find two real vectors, $\vec{\xi}_a$ and $\vec{\xi}_b$, that are linear combinations of $\vec{\xi}_1$ and $\vec{\xi}_2$, and are such that $\{\vec{\xi}_a, \vec{\xi}_b, \vec{\xi}_3\}$ are a normalized orthogonal set of vectors. This is the basis that we use as the axes for our new coordinates.

(b) [2 points] Write the relation between components in the basis $\{\vec{\xi}_a, \vec{\xi}_b, \vec{\xi}_3\}$ and $\{\vec{\xi}_1, \vec{\xi}_2, \vec{\xi}_3\}$ as a matrix W, and show that W is unitary.

(c) [4 points] Consider the new coordinates whose components are described by the transformation $\vec{u} = W\vec{s} = WX^\dagger\vec{r}$. Write the rotation $\vec{r}\,' = U\vec{r}$ as a relation $\vec{u}\,' = \widetilde{U}\vec{u}$, and show that the matrix $\widetilde{U}$ has the form of a standard rotation matrix with rotation angle Φ.

Note: $\mathrm{Tr}(U) = \mathrm{Tr}(\widetilde{U}) = 1 + 2\cos\Phi$, so given a matrix U we can easily determine Φ without going through transformations.

2. **Foucault Pendulum and the Coriolis Effect** [13 points]

Consider a pendulum consisting of a long massless rod of length ℓ attached to a mass m. The pendulum is hung in a tower that is at latitude λ on the earth's surface, so it is natural to describe its motion with coordinates fixed to the rotating Earth. Let ω be the Earth's angular velocity. Use the spherical coordinates (r, θ, ϕ) shown in the figure to investigate the Coriolis force. Here $\hat{z}$ is perpendicular to the Earth's surface and $\hat{y}$ is tangent to a circle of constant longitude that passes through the north pole.

a) [9 points] The velocity is given in terms of $\vec{v}$ in the rotating frame by $\vec{v} + \vec{\omega} \times (R_e\hat{z} + \vec{r})$, so

$$L = \frac{m}{2}\left[\vec{v} + \vec{\omega} \times (\vec{r} + R_e\hat{z})\right]^2 - V\,,$$

with R_e the radius of the earth, and V the potential energy due to gravity near the earth's surface (we neglect air resistance). Writing everything in terms of the variables θ and ϕ, and the fixed angle λ, derive the equations of motion for the pendulum. From the start you should only keep terms up to first order in ω. You can also drop the term $\propto \omega R_e$ since it is a total time derivative.

b) [4 points] Since ℓ is large, consider the small angle approximation for θ and simplify your equations of motion from a). Demonstrate that the pendulum undergoes precession with $\dot{\phi} = \omega \sin\lambda$.

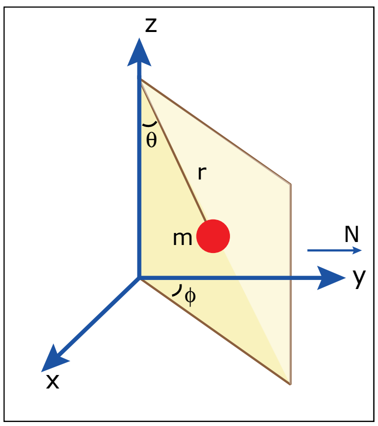


Image by MIT OpenCourseWare.

3. **Angular Velocity with Euler Angles** [9 points]

(a) [2 points] Show that the components of angular velocity along the body axes (x', y', z') are given in terms of Euler angles by

$$\omega_{x'} = \dot{\phi}\,\sin\theta\,\sin\psi + \dot{\theta}\,\cos\psi\,,$$
$$\omega_{y'} = \dot{\phi}\,\sin\theta\,\cos\psi - \dot{\theta}\,\sin\psi\,,$$
$$\omega_{z'} = \dot{\phi}\,\cos\theta + \dot{\psi}\,.$$

This is done in the text! I am asking you to go through the steps to ensure that you understand the calculation.

(b) [4 points] Show that the components of angular velocity along the fixed space set of axes, the inertial frame (x, y, z), are given in terms of the Euler angles by

$$\omega_x = \dot{\theta}\,\cos\phi + \dot{\psi}\,\sin\theta\,\sin\phi\,,$$
$$\omega_y = \dot{\theta}\,\sin\phi - \dot{\psi}\,\sin\theta\,\cos\phi\,,$$
$$\omega_z = \dot{\psi}\,\cos\theta + \dot{\phi}\,.$$

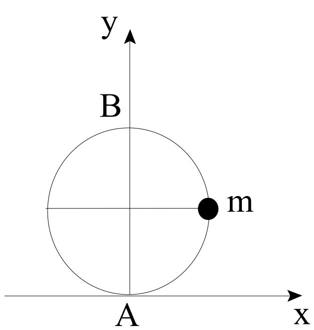
This problem is Goldstein Ch.4#14. You may use results given in Goldstein.

(c) [3 points] Using the generalized coordinate ψ, derive an Euler equation of motion using the Euler-Lagrange equation. Use the form with generalized forces Q_j:

$$\frac{d}{dt}\frac{\partial T}{\partial \dot{q}_j} - \frac{\partial T}{\partial q_j} = Q_j \, .$$

4. **Point Mass on a Disk** [12 points]

A thin uniform disk of radius R and mass M lies in the x-y plane, and has a point mass $m = 3M/8$ attached on its edge. (There is no gravity in this problem.)



(a) [4 points] Find the moment of inertia tensor of the disk about its center (ignoring the mass m). Then find the moment of inertia tensor of the combined system of the disk and point mass about the point A in the figure.

(b) [4 points] Find the principal moments of intertia and the principal axes about A. (Recall that you may use mathematica or matlab. If you do then you should still write out intermediate steps.)

(c) [4 points] The disk is constrained to rotate about the y-axis with angular velocity ω by pivots at A and B. What is the angular momentum about A as a function of time?

5. **A Rolling Cone** [16 points] (Adapted from Goldstein Ch.5 #17)

A uniform right circular cone of height h, half-angle α, and density ρ rolls on its side without slipping on a uniform horizontal plane. It returns to its original position in a time τ.

(a) [5 points] Find the CM of the cone. Find the moment of inertia tensor for body axes centered on the tip with the y'-axis going through the CM.

(b) [2 points] What is the moment of inertia tensor if we move our axes in (a) so they are centered on the CM?

(c) [4 points] Now assume the cone rolls on a fixed plane. Pick a new set of body axes (x, y, z) such that the z-axis is perpendicular to the plane, the y-axis coincides with the instantaneous line of contact, and the origin is the tip of the cone. Find the moment of inertia tensor for these axes.

(d) [5 points] Find the kinetic energy of the rolling cone. [There are two ways you could answer this, one uses your results from (b) and one uses those from (c).]