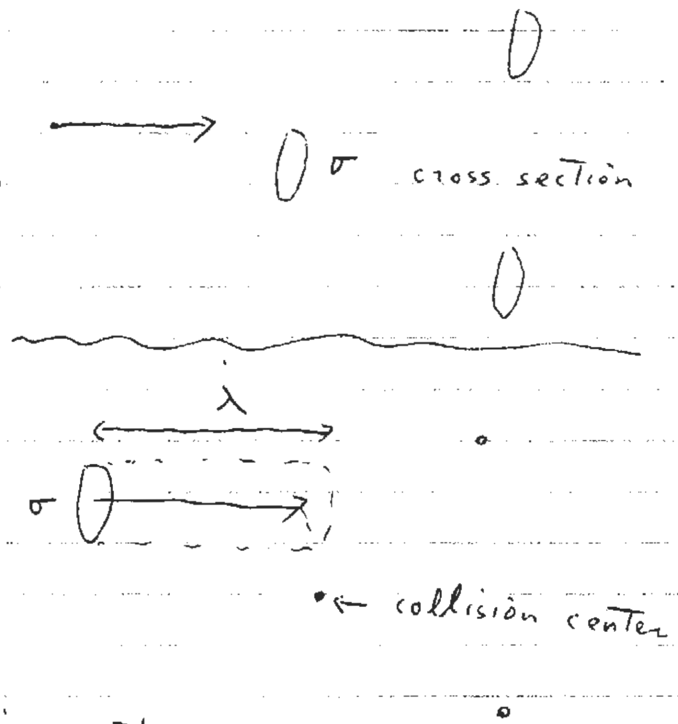
Transport:

Two limits Collisionless and hydrodynamic

Mean-free path.

σ scattering crossection

$\lambda\sigma$: contain one collision center on average.

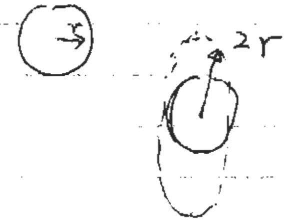


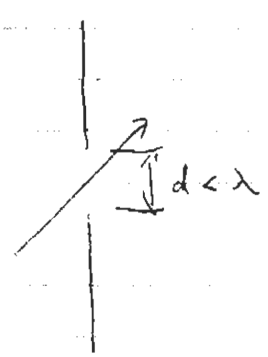
$$\lambda = \frac{1}{\sigma n}$$

n: density of collision center

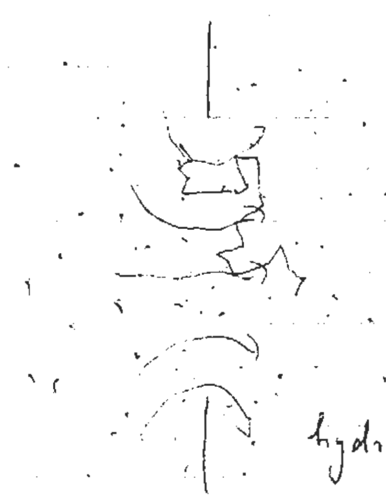
Scattering crossection of two hard balls

$$\sigma = \pi (2r)^2 = 4\pi r^2$$





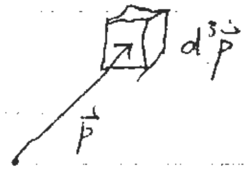
effusion



hydrodynamic flow

Effusion

distribution of momentum



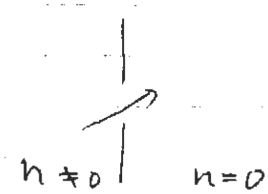
$f(\vec{p})\, d^3\vec{p}$ = density of particles with momentum in d^3p

$$\boxed{f(\vec{p}) = C\, e^{-\beta \vec{p}^2/2m}}$$ Boltzmann distribution

$\int d^3p \; f(\vec{p}) = n$ ← total density of particles.

$$\Rightarrow \boxed{C = n\,(2\pi m k_B T)^{-3/2}}$$

Flux of effusion: (# of particle per unit area per second)



$$I = \int_{v_x > 0} v_x\, f(p)\, d^3\vec{p}$$

$$= C \int_{p_x > 0} \frac{p_x}{m}\, e^{-\beta(p_x^2 + p_y^2 + p_z^2)/2m}\, dp_x\, dp_y\, dp_z$$

$$= n\sqrt{\frac{k_B T}{2\pi m}}$$

$I \propto n\, T^{1/2}\, m^{-1/2}$

Why?

Physical picture

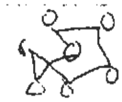
Non-viscous hydrodynamics

- Three equations

$\lambda/\ell \to 0$ $\quad\lambda$: mean-free path

$\quad\ell$: scale of interest

particles are "caged" by their neighbors



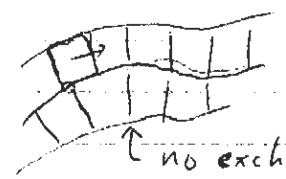
picture

mass density : $\rho(\vec{x},t) = m\, n(\vec{x},t)$

mass current : $\rho\,\vec{u}$

↑ average velocity of particles

~ velocity of gas/fluid



↑ no exchange of particles, heat, ...

① Mass conservation

$$\boxed{\frac{\partial \rho}{\partial t} + \nabla\cdot(\rho\vec{u}) = 0}$$ continuity equation

$$\Rightarrow \frac{d}{dt}\int_V d^3\vec{x}\,\rho + \oint_{\partial V} d\vec{S}\cdot\rho\vec{u} = 0$$

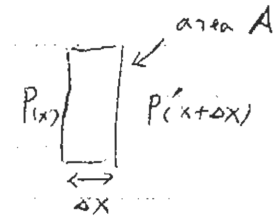
② Newton's law.

$$\frac{d\vec{u}}{dt} = \frac{1}{m}\vec{f}$$ ← force on the particle

↑ if we follow one particle

$$\frac{d\vec{u}}{dt} = ? \qquad \frac{d\vec{u}}{dt} = \underbrace{\left(\frac{\partial}{\partial t} + \vec{u}\cdot\nabla\right)}_{\equiv \frac{d}{dt}}\vec{u}(\vec{x},t)$$

↑ velocity field



$$F_x = A\,(P(x) - P(x+\Delta x)]$$

$$= -\Delta x\, A\ \partial_x P \qquad \text{on } N \text{ particle}$$

on one particle $\quad f_x = \dfrac{F_x}{N} = -\dfrac{1}{n}\partial_x P$

$$\left(\frac{\partial}{\partial t} + \vec{u}\cdot\nabla\right)\vec{u} = -\frac{1}{mn}\vec{\nabla} P + \frac{1}{m}\vec{f}_{ex}$$

or $\boxed{\rho\left(\dfrac{\partial}{\partial t} + \vec{u}\cdot\nabla\right)\vec{u} + \nabla P = n\vec{f}_{ex}}$ Euler's equ.

force per particle

(3) Adiabatic condition conservation of entropy per particle — s :

. s does not change with the flow

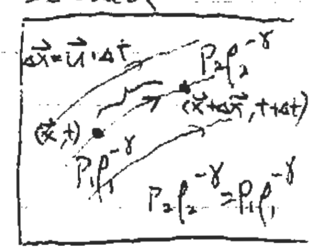
$$S = Nk\left[\frac{5}{2} - \ln(n\lambda_T^3)\right], \quad s = \frac{S}{N}, \ \lambda = \sqrt{\frac{2\pi\hbar^2}{mk_B T}}$$

$\Rightarrow \quad nT^{-3/2}$ is conserved

$P = nk_BT \quad$ or $\quad T = \dfrac{P}{nk_B}$

$\Rightarrow \quad n\left(\dfrac{n}{P}\right)^{3/2} = n^{5/2}P^{-3/2}$ is conserved

or $\boxed{P\rho^{-5/3} \quad \text{is conserved}}$



$$\boxed{\left(\frac{\partial}{\partial t} + \vec{u}\cdot\nabla\right)(P\rho^{-\gamma}) = 0} \qquad \gamma = \frac{5}{3}$$

Adiabatic condition

5 unknown ρ, P, $\vec{u}$ & 5 equations.

★ Linearized equation

$$\rho = \rho_0 + \delta\rho$$

$$P = P_0 + \delta P$$

$$\vec{u} = 0 + \vec{u}$$

$$\boxed{\begin{aligned} &\frac{\partial \delta\rho}{\partial t} + \rho_0 \nabla \cdot \vec{u} = 0 \\ &\rho_0 \frac{\partial \vec{u}}{\partial t} + \nabla \delta P = 0 \end{aligned}}$$

$$\frac{\partial}{\partial t}(P\rho^{-\gamma}) = \rho_0^{-\gamma} \frac{\partial}{\partial t}\delta P - \gamma P_0 \rho_0^{-\gamma-1} \frac{\partial}{\partial t}\delta\rho = 0$$

$$\boxed{\frac{\partial}{\partial t}\delta P - \gamma \frac{P_0}{\rho_0} \frac{\partial}{\partial t}\delta\rho = 0}$$

★ Sound waves.

$$\hookrightarrow \boxed{\frac{\delta P}{P_0} = \gamma \frac{\delta\rho}{\rho_0}}$$

$$\frac{\partial^2 \delta\rho}{\partial t^2} + \rho_0 \nabla \cdot \frac{\partial \vec{u}}{\partial t} = 0$$

$$\frac{\partial^2 \delta\rho}{\partial t^2} - \nabla^2 \delta P = 0$$

$$\frac{\partial^2 \delta\rho}{\partial t^2} - \nabla^2 \delta\rho \, \gamma \frac{P_0}{\rho_0} = 0$$

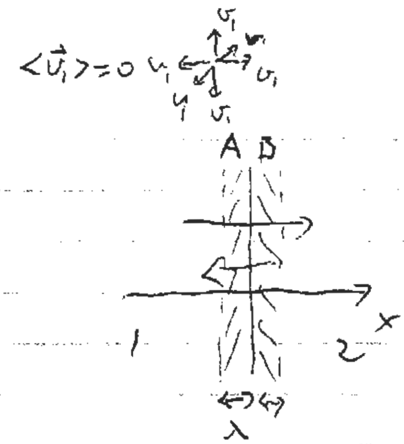
$$\Rightarrow \frac{\partial^2 \delta\rho}{\partial t^2} - c^2 \nabla^2 \delta\rho = 0$$

$$P = n k_B T$$

$$\boxed{c = \sqrt{\frac{\gamma P_0}{\rho_0}} = \sqrt{\frac{\gamma k_B T}{m}}}$$

$$\gamma = \frac{5}{3}$$

$c \sim$ thermal velocity of particle



mean free path

★ Diffusion estimate

Net flux : 6 directions

$$j_x = \frac{1}{6}\bar{v}_1 n_1 - \frac{1}{6}\bar{v}_2 n_2$$

$$= \frac{1}{6}\bar{v}\,(n_1 - n_2) \qquad \text{assume } \bar{v} \text{ is the same}$$

what $n_1 - n_2$?

n_1 density of particle coming from region A

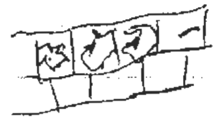
n_2 for region B

$$n_1 - n_2 = -\,\partial_x n \cdot \lambda$$

$$\boxed{j_x = -\frac{1}{6}\bar{v}\lambda\,\partial_x n}$$

diffusion flux.

in hydrodynamical limit we set $\lambda = 0$ no diffusion



in general

$$\boxed{\vec{j} = -D\,\nabla n}$$

$$D \simeq \frac{\lambda \bar{v}}{6}$$

Continuity equation (with diffusion)

$$\frac{\partial n}{\partial t} + \nabla \cdot \vec{j} = 0$$

$$\vec{j} = n\vec{u} - D\nabla n$$

$$\Rightarrow \quad \boxed{\frac{\partial n}{\partial t} + \nabla \cdot n\vec{u} - D\nabla^2 n = 0}$$

or $$\boxed{\frac{\partial \rho}{\partial t} + \nabla \cdot \rho \vec{u} - D\nabla^2 \rho = 0}$$

diffusion equation (set $\vec{u} = 0$)

$$\frac{\partial n}{\partial t} - D\nabla^2 n = 0$$

if $n(t=0, \vec{x}) = N\delta(\vec{x})$ (N particle at $\vec{x} = 0$)

in k-space

$$\frac{\partial \tilde{n}}{\partial t} + Dk^2 \tilde{n} = 0$$

$$\tilde{n}(t, k) = e^{-Dk^2 t}$$

$$\tilde{n}(t, \vec{x}) = \int \frac{d^3k}{(2\pi)^3} e^{-Dk^2 t} e^{i\vec{k}\cdot\vec{x}}$$

$$= \int \frac{d^3k}{(2\pi)^3} e^{-Dt\left(\vec{k} - \frac{i}{2}\frac{\vec{x}}{Dt}\right)^2} e^{-\frac{1}{4}\frac{\vec{x}^2}{Dt}}$$

$$\Rightarrow \boxed{n(\vec{x},t) = \frac{N}{(4\pi D t)^{3/2}} e^{-\frac{|\vec{x}|^2}{4Dt}}} \quad \circledast$$

$$\int dx\, e^{-x^2} = \sqrt{\pi}$$

$$\int dx\, x^2 e^{-x^2} = \frac{1}{2}\sqrt{\pi}$$

$$\langle x^2 \rangle = \int d^3x\ \vec{x}^2\ \frac{e^{-|x|^2/4Dt}}{(4\pi D t)^{3/2}}$$

$$= 4Dt \times \frac{3}{2}$$

$$= 6Dt$$

$$\boxed{\bar{x} \equiv \sqrt{\langle x^2 \rangle} = \sqrt{6Dt}}$$

Random walk & diffusion

each step λ

N step later

$$\vec{x} = \Delta\vec{x}_1 + \Delta\vec{x}_2 + \cdots \qquad |\Delta\vec{x}_1| = |\Delta\vec{x}_2| = \cdots = \lambda$$

Distribution of $\vec{x}$

$$n(\vec{x}, N) = \frac{1}{(\pi\alpha)^{3/2}} e^{-\vec{x}^2/\alpha} \qquad \alpha = ?$$

$$\langle \vec{x}^2 \rangle = \Delta\vec{x}_1^2 + \Delta\vec{x}_2^2 \cdots$$

$$= N\lambda^2 \Rightarrow \boxed{\bar{x} = \lambda\sqrt{N}}$$

$$\boxed{D = \frac{\lambda^2}{6\tau} = \frac{\lambda v}{6}}$$

$$= \int d^3x\ \vec{x}^2\ n = \frac{3}{2}\alpha \Rightarrow \alpha = \frac{2N\lambda^2}{3}$$

$\Rightarrow$ See $\circledast$

$$\boxed{n(\vec{x}) = \frac{1}{(\pi \frac{2N\lambda^2}{3})^{3/2}} e^{-\vec{x}^2/\frac{2N\lambda^2}{3}} = \frac{e^{-\vec{x}^2/\frac{2\lambda^2}{3\tau}t}}{(\pi \frac{2\lambda^2}{3\tau}t)^{3/2}}}$$

Damping of sound wave due to diffusion

Linearized :

$$\frac{\partial \delta\rho}{\partial t} + \rho_0 \nabla\cdot\vec{u} - D\nabla^2 \delta\rho = 0$$

$$\rho_0 \frac{\partial \vec{u}}{\partial t} + \nabla \delta p = 0$$

$$\frac{\partial^2 \delta\rho}{\partial t^2} + \rho_0 \nabla\cdot\frac{\partial \vec{u}}{\partial t} - D\nabla^2 \frac{\partial}{\partial t}\delta\rho = 0$$

$$\frac{\partial^2 \delta\rho}{\partial t^2} - c^2\nabla^2\delta\rho - D\nabla^2\frac{\partial}{\partial t}\delta\rho = 0$$

$$\delta\rho = A e^{i(kx - \omega t)}$$

$$-\omega^2 + c^2k^2 + Dk^2(-i\omega) = 0$$

small D

$$\omega^2 = c^2k^2 - iDk^3c = c^2k^2\left(1 - i\frac{Dk}{c}\right)$$

$$\omega = ck\left(1 - \frac{1}{2}i\frac{Dk}{c}\right)$$

$$\delta\rho = A e^{i(kx - ckt)} e^{-t/\tau}$$

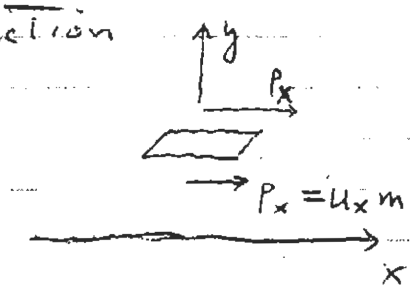
$$\boxed{\tau = \frac{2}{Dk^2}} = \frac{12}{\lambda \underset{\sim c}{v} k^2} \sim \frac{1}{\omega}\frac{1}{\lambda k}$$

A sound wave can oscillate $\frac{1}{\lambda k}$ times

$$\frac{1}{\tau} = \frac{Dk^2}{2}$$ damping rate

damping coefficient

Viscosity — momentum conduction
(due to diffusion)



Flux of P_x momentum

= Force per area F_x

$$F_x = \frac{1}{6} n\bar{v}\,(u_{1x}\, m - u_{2x} m)$$

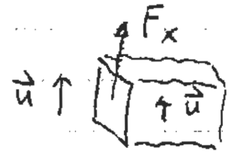
$$= -\frac{1}{6} n m \bar{v} \lambda \frac{\partial u_x}{\partial y}$$

$$F_x = -\nu \frac{\partial u_x}{\partial y}$$ = flux of x momentum in y-direction

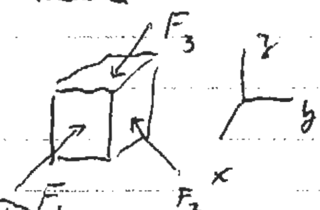
= Force in x-direction on a surface normal to y.

$$\boxed{\nu = \frac{1}{6} n \bar{v} m \lambda = \frac{\sqrt{2 m k_B T}}{3\sigma}}$$

independent of n.



Pressure tensor P_{ij}:



$$\vec{F}_1 = (F_{1x}, F_{1y}, F_{1z}) = A(P_{11}, P_{12}, P_{13})$$

$$\vec{F}_2 = A(P_{21}\ P_{22}\ P_{23})$$

$$\vec{F}_3 = A\ (P_{31}\ P_{32}\ P_{33})$$

$\boxed{\vec{F}_1 = \text{force on surface normal to } \hat{x}}$

$$P_{ij} = \underbrace{\delta_{ij} P}_{\text{usual pressure}} + \underbrace{P'_{ij}}_{\text{traceless part}} \quad \& \quad P'_{ij} = -\nu\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} - \frac{2}{3}\delta_{ij}\nabla\cdot\vec{u}\right)$$

Navier – Stokes equation

$$\underbrace{\Delta V \rho}_{\text{mass}} \underbrace{\left(\frac{\partial}{\partial t} + \vec{u}\cdot\nabla\right) u_i}_{\frac{d u_i}{dt}} = \text{force in } i\text{-direction}$$

$$= + A \sum_j \left(P_{ij}(x+\Delta x_j) + P_{ij}(x)\right)$$

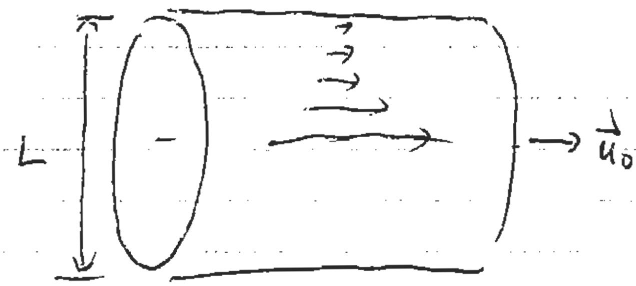
$$\Rightarrow \rho\left(\frac{\partial}{\partial t} + \vec{u}\cdot\nabla\right) u_i + \frac{\partial P_{ij}}{\partial x_j} = 0$$

$$\Rightarrow \boxed{\rho\left(\frac{\partial}{\partial t} + \vec{u}\cdot\nabla\right)\vec{u} + \nabla\cdot\left(p - \frac{\nu}{3}\nabla\cdot\vec{u}\right) - \nu\nabla^2\vec{u} = 0}$$

$$= \vec{f}_{ex}$$

Any quantity can be set to 1 by changing the unit.

$$\boxed{\begin{array}{l} \dfrac{\partial\rho}{\partial t} + \nabla\cdot(\rho\vec{u}) = 0 \\ \left(\dfrac{\partial}{\partial t} + \vec{u}\cdot\nabla\right)(p\rho^{-\gamma}) = 0 \end{array}}$$



The above three equations determine the flow in the pipe. quantities characters the effect of viscosity. Inputs: $\boxed{u_0,\ L,\ \nu,\ \rho}$

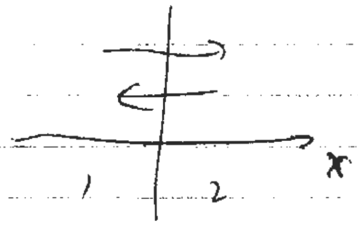
$$\boxed{\begin{array}{ll} R \gg 1 & \text{Turbulent} \\ R \ll 1 & \text{streamline} \end{array}}$$

$$[\nu] = \frac{\text{force/area}}{\text{velocity/length}} = \frac{m v}{t L^2}\,\frac{L}{v} = \frac{m}{t L}$$

$$R = \frac{\rho L u_0}{\nu} \quad \text{dimensionless Reynolds number}$$

Heat conduction – energy exchange without particle exchange.

$$j_x = \frac{1}{6}\left(n_1\bar{v}_1 - n_2\bar{v}_2\right) = 0$$



$$n_1 \approx n_2 \qquad \bar{v}_1 \approx \bar{v}_2$$

$$q_x = \frac{1}{6} n\bar{v} \left(l\frac{k_B T_1}{2} - l\frac{k_B T_2}{2} \right)$$

l – number of degrees of freedom

$l = 3$ for point particle

$$T_1 - T_2 = -\partial_x T\, \lambda$$

λ – mean free path.

$$q_x = -\frac{1}{12}\lambda l n\bar{v} k_B \partial_x T$$

$$\boxed{\vec{q} = -K\, \nabla T}$$

$$K = \frac{l n \bar{v} k_B \lambda}{12}$$

$$= \frac{1}{6} n\bar{v}\lambda c_v$$

c_v – specific heat per particle

heat conductance

$$c_v = \frac{l k_B}{2}$$

Using $\bar{v} = \sqrt{2k_BT/m}$ $\qquad \lambda = 1/n\sigma$

σ – scattering crossection

$$\boxed{K = \frac{c_v}{3\sigma}\sqrt{\frac{2k_BT}{m}}}$$ independent of n

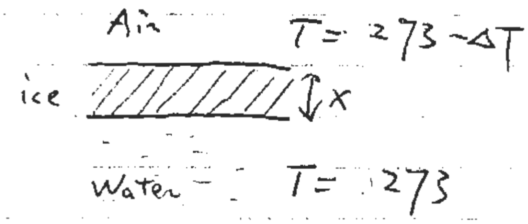
Energy conservation $\frac{\partial u}{\partial t} + \nabla\cdot\vec{q} = 0$

heat conductance equation

since $\vec{q} = -K\nabla T \qquad u = n c_v T \Rightarrow$

$$\boxed{\frac{\partial T}{\partial t} - \frac{K}{n c_v}\nabla^2 T = 0}$$

with heat source $$\boxed{n c_v \frac{\partial T}{\partial t} - K\nabla^2 T = W}$$



* Speed of freezing

– heat tranport per unit area per unit time $q = K \frac{\Delta T}{x}$

– heat generated by freezing

ΔS – change of entropy per atom

$l = \Delta S\, T$ – latent heat per atom.

heat generated per unit area per unit time

$$\frac{dx}{dt}\, \underbrace{n}_{\text{number density}} \cdot l = K \frac{\Delta T}{x}$$

$$\Rightarrow \frac{dx}{dt} = \frac{K \Delta T}{n\, l} \frac{1}{x}$$

$$\Rightarrow x^2 = 2 \frac{K \Delta T}{n l} t \quad \text{or} \quad \boxed{x = \sqrt{\frac{2 K \Delta T}{n\, l}} \sqrt{t}}$$

Numerical estimate

ΔS : Ideal gas $S = k_B \left(\frac{5}{2} + \ln \frac{v}{\lambda^3} \right)$ ← Volum per atom

assume $\boxed{v_{water} = 2 v_{ice}}$ $\Rightarrow$ $\boxed{\Delta S = k_B}$

$\boxed{l = k_B T_0}$ $\qquad c_V \sim k_B$

$$K = \frac{c_V}{3\sigma} \sqrt{\frac{2 k_B T}{m}} = \frac{k_B}{\sigma} \sqrt{\frac{k_B T}{m}}$$

a: lattice constant

$$\sqrt{\frac{2 K \Delta T}{n\, l}} \sim \sqrt{\frac{\frac{k_B}{\sigma} \sqrt{\frac{k_B T}{m}}\, \Delta T}{n\, k_B T_0}} = \sqrt{\frac{\bar{v}\, \Delta T}{\sigma\, n\, T_0}} \sim \sqrt{a \bar{v} \frac{\Delta T}{T_0}}$$

$a \sim 3 \mathring{A}$ $\quad \bar{v} = 300\,m/s$ $\quad \sqrt{a \bar{v}} = 0.3\,mm \frac{1}{\sqrt{sec}}$ $\quad \boxed{x = 0.3\,mm \sqrt{\frac{\Delta T}{T}} \sqrt{\frac{t}{sec}}}$

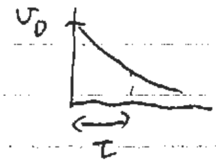
Transport in a metal

☆ **Drude model:**

$$\Delta\langle u\rangle = \underbrace{\left(1-\frac{\Delta t}{\tau}\right)}_{\text{fraction with no collision}}\langle u\rangle + \frac{\Delta t}{\tau}\langle 0\rangle$$

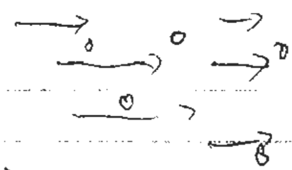
$$\frac{du}{dt} = \underbrace{-\frac{1}{\tau}}_{\text{collision}}u \qquad \leftarrow u \text{: average velocity}$$

$$v = v_0 e^{-t/\tau}$$



$\tau \sim$ mean-free "time"

$\tau = \lambda/\bar{v}$ ← from scattering with impurity



with external electric field

$$\frac{du}{dt} = -\frac{1}{\tau}u + \frac{e}{m}E$$

$j = enu$ — charge current density.

$$\boxed{j + \tau\frac{dj}{dt} = \frac{e^2 n\tau}{m}E}$$

steady state $\boxed{\vec{j} = \sigma\vec{E}, \quad \sigma = \frac{e^2 n\tau}{m}}$

Ohm's law.

Temperature dependence of σ.

Only τ depen on T.

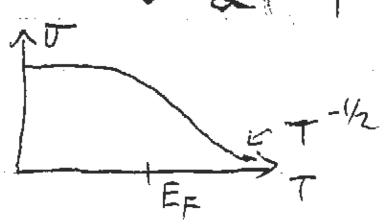
$\tau \sim \lambda/\bar{v}$

λ: no T dependence, impurity density is fixed

$\boxed{\bar{v} \simeq \sqrt{\frac{k_B T}{m}}}$ ✗ $\Rightarrow \tau \propto T^{-1/2}$ or $\sigma \propto T^{-1/2}$ ✗

$\bar{v} \simeq \sqrt{\frac{E_F}{m}}$ ✓ $\simeq \frac{\hbar n^{1/3}}{m}$

$$\boxed{\sigma \propto T^0} \checkmark$$



* Hall effect

$$\frac{d\vec{u}}{dt} = -\frac{1}{\tau}\vec{u} + \frac{e}{m}\vec{E} + \frac{e}{mc}\vec{u}\times\vec{B} \quad (\text{cgs})$$

$$\vec{j} = en\vec{u}$$

$$\frac{d\vec{j}}{dt} = -\frac{1}{\tau}\vec{j} + \frac{e^2n}{m}\vec{E} + \frac{e}{mc}\vec{j}\times\vec{B}$$

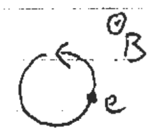
steady flow

$$\boxed{\vec{j} = \frac{e^2n\tau}{m}\vec{E} + \frac{e\tau}{mc}\vec{j}\times\vec{B}}$$

Let $\vec{B} = (0, 0, B)$ $\quad \vec{j}\times\vec{B} = (j_y B, -j_x B, 0)$

$$j_z = \frac{en\tau}{m} E_z$$

$\omega_c = eB/mc$



cyclotron frequency

$$\begin{pmatrix} j_x \\ j_y \end{pmatrix} = \underbrace{\sigma_0}_{\frac{e^2n\tau}{m}} \begin{pmatrix} E_x \\ E_y \end{pmatrix} + \begin{pmatrix} 0 & \omega_c\tau \\ -\omega_c\tau & 0 \end{pmatrix}\begin{pmatrix} j_x \\ j_y \end{pmatrix}$$

or

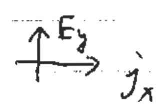
$$\boxed{\begin{pmatrix} E_x \\ E_y \end{pmatrix} = \sigma_0^{-1}\begin{pmatrix} 1 & -\omega_c\tau \\ \omega_c\tau & 1 \end{pmatrix}\begin{pmatrix} j_x \\ j_y \end{pmatrix}} = \begin{pmatrix} \rho_{xx} & \rho_{xy} \\ \rho_{yx} & \rho_{yy} \end{pmatrix}\begin{pmatrix} j_x \\ j_y \end{pmatrix}$$

$$|\rho_{xy}| = |\rho_{yx}| = \frac{\omega_c\tau}{\sigma_0} = \frac{eB\cancel{\tau}}{\cancel{m}c}\,\frac{\cancel{m}}{e^2n\cancel{\tau}} = \frac{B}{ecn}$$

$$|\rho_{xy}| = \frac{B}{enc} = R_H B \qquad \boxed{R_H = \frac{1}{enc}}$$

meaning : if $\begin{pmatrix} j_x \\ j_y \end{pmatrix} = \begin{pmatrix} j_x \\ 0 \end{pmatrix}$ $\quad \begin{pmatrix} E_x \\ E_y \end{pmatrix} = \begin{pmatrix} \rho_{xx} \\ \rho_{xy} \end{pmatrix} j_x$

$$\boxed{E_y = \rho_{xy}\, j_x = R_H B j_x}$$



Boltzmann equation (fluid in $(\vec{x}, \vec{k})$)

$g(\vec{r}, \vec{k}, t)$: $\quad dN = g(\vec{r}, \vec{k}, t) \dfrac{d^3\vec{r}\, d^3\vec{k}}{(2\pi)^3}$

~~number of particle in $d^3\vec{r}\, d^3\vec{k}$~~

number of particle in a $\vec{k}$ level

Equilibrium distribution

$$g_0 = \frac{1}{e^{\beta(\epsilon_k - \mu)} + 1} \qquad \text{fermion}$$

$$g_0 = \frac{1}{e^{\beta(\epsilon_k - \mu)} - 1} \qquad \text{boson}$$

$$g_0 = e^{-\beta(\epsilon_k - \mu)} \qquad \text{classical gas}$$

If $g(\vec{r}, \vec{k}, t) \neq g_0$ non-equilibrium distribution

Diffussionless motion (hydrodynamic motion)

$$dN(t) = dN(t - dt)$$

$$g(\vec{r}, \vec{k}, t) \underbrace{d^3r(t)\, d^3k(t)}_{d^3r\, d^3k} = g(\vec{r} - \vec{v}_{(k)} dt, \vec{k} - \vec{F}\frac{dt}{\hbar}, t - dt) \times \underbrace{d^3\vec{r}(t-dt)\, d^3\vec{k}(t-dt)}_{d^3r'\, d^3k'}$$

Liouville's Theorem

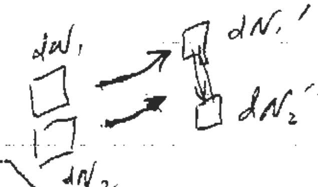
$$d^3r\, d^3k = d^3\vec{r}'\, d^3\vec{k}'$$

$$g(\vec r, k, t) = g\left(\vec r - \vec v_k\, dt,\ \vec k - \vec F \frac{dt}{\hbar},\ t - dt\right)$$

$$\boxed{\frac{\partial g}{\partial t} + \vec v \cdot \frac{\partial}{\partial \vec r} g + \frac{1}{\hbar}\vec F \cdot \frac{\partial}{\partial \vec k} g = 0}$$

no diffusion (collission effect)

relaxation-time approximation



$$\boxed{\frac{\partial g}{\partial t} + \vec v \cdot \frac{\partial}{\partial \vec r} g + \frac{1}{\hbar}\vec F \cdot \frac{\partial}{\partial \vec k} g = -\frac{1}{\tau}\left(\underset{\text{out}}{g} - \underset{\text{in}}{g_0}\right)}$$

$-\frac{1}{\tau} g$ out-Term

$-\frac{1}{\tau} g_0$ in-Term

Let: $\delta g = g - g_0$

Steady state:

$$\delta g = -\tau \vec v_{(k)} \frac{\partial}{\partial \vec r} g_0 - \frac{\tau}{\hbar} \vec F \cdot \frac{\partial}{\partial \vec k} g_0$$

$$- \tau \vec v \frac{\partial}{\partial \vec r} \delta g - \frac{\tau}{\hbar} \vec F \frac{\partial}{\partial \vec k} \delta g \quad \leftarrow \quad O\left(\frac{\partial}{\partial \vec r} g_0, \vec F\right)^2$$

Assume $\frac{\partial}{\partial \vec r} g_0$ and $\vec F$ is small

$$\delta g = O\left(\frac{\partial}{\partial \vec r} g_0, \vec F\right)$$

$$\frac{\partial f}{\partial \vec r} = T \frac{\partial f}{\partial \epsilon} \frac{\partial \beta(\epsilon - \mu)}{\partial \vec r}$$

$$= + \frac{\partial f}{\partial \epsilon}\left(-\nabla \mu - \frac{\epsilon - \mu}{T} \nabla T\right)$$

$$\boxed{\delta g = -\tau \vec v_{(k)} \frac{\partial}{\partial \vec r} g_0 - \frac{\tau}{\hbar} \vec F \cdot \frac{\partial}{\partial \vec k} g_0}$$

$g_0(T(\vec r), \mu(\vec r), \vec k)$ for fermion $g_0 = f = \frac{1}{e^{\beta(\epsilon - \mu)} + 1}$

For fermion $g_0 = f = \dfrac{1}{e^{\beta(\vec{r})[\epsilon_{\vec{k}} - \mu(\vec{r})]}+1}$

$$\frac{\partial f}{\partial \vec{r}} = \frac{1}{\beta}\frac{\partial f}{\partial \epsilon} \frac{\partial[\beta(\epsilon-\mu)]}{\partial \vec{r}}$$

$$= T\frac{\partial f}{\partial \epsilon}\left(-\beta\frac{\partial \mu}{\partial \vec{r}} - \frac{\epsilon-\mu}{T^2}\frac{\partial T}{\partial \vec{r}}\right)$$

$$= -\frac{\partial f}{\partial \epsilon}\left(\nabla\mu + \frac{\epsilon-\mu}{T}\nabla T\right)$$

$$\frac{\partial f}{\partial \vec{k}} = \frac{\partial f}{\partial \epsilon}\frac{\partial \epsilon}{\partial \vec{k}}$$

$$= \hbar \vec{v}_k \frac{\partial f}{\partial \epsilon}$$

$$\boxed{\delta g_{(\vec{r},\vec{k})} = \left(-\frac{\partial f}{\partial \epsilon}\right) \tau \vec{v}\cdot\left[-\nabla\mu + \vec{F} - \frac{\epsilon-\mu}{T}\nabla T\right]}$$

$$\boxed{\delta g(\vec{r},\vec{k}) = \left(-\frac{\partial f}{\partial \epsilon}\right)_{(\vec{r},\vec{k})} \tau \vec{v}_{\vec{k}}\cdot\left[-\nabla\mu(\vec{r}) + \vec{F}(\vec{r},\vec{k}) - \frac{\epsilon_{\vec{k}}-\mu(\vec{r})}{T(\vec{r})}\nabla T\right]}$$

$$\boxed{\vec{F}(\vec{r},\vec{k}) = e\vec{E}(\vec{r}) + \frac{e}{c}\vec{v}(\vec{k})\times\vec{B}(\vec{r})}$$

One we know $\vec{E}(\vec{r})$, $\vec{B}(\vec{r})$, $T(\vec{r})$, $\mu(\vec{r})$

we know $g(\vec{r},\vec{k}) = g_0 + \delta g$

we can calculate any thing

Also works for Boson gas if we replace

$f = \dfrac{1}{e^{\beta(\epsilon-\mu)}+1}$ by $f_b = \dfrac{1}{e^{\beta(\epsilon-\mu)}-1}$

or classical gas by $f_c = \dfrac{1}{e^{\beta(\epsilon-\mu)}}$

Application : electric current

$\vec{B}=0$

$$g = g_0 + \underbrace{\tau\left(-\frac{\partial f}{\partial \epsilon}\right)\vec{v}_{\vec{k}}\cdot\left[+e\vec{\mathcal{E}} + \frac{\epsilon-\mu}{T}(-\nabla T)\right]}_{\delta g}$$

$$\boxed{\vec{\mathcal{E}} \equiv \vec{E} - \frac{\nabla\mu}{e}}$$

electric current

$$j_x = \int \frac{d^3k}{(2\pi)^3}\, e\, v_x(\vec{k})\, g(\vec{k}) = \int \frac{d^3\vec{k}}{(2\pi)^3}\, e\, v_x \cdot \delta g$$

$$= \int \frac{d^3k}{(2\pi)^3}\, \tau e^2 \left(-\frac{\partial f}{\partial \epsilon}\right) v_x\, \vec{v}\cdot\vec{\mathcal{E}}$$

$$+ \int \frac{d^3k}{(2\pi)^3}\, e\tau \left(-\frac{\partial f}{\partial \epsilon}\right) \frac{\epsilon-\mu}{T}\, v_x\, \vec{v}\cdot(-\nabla T)$$

$$= L^{11}\mathcal{E}_x + L^{12}(-\partial_x T)$$

$d^3k = 4\pi k^2 dk$

$= 2\pi k\, dk^2$

$= 2\pi k \frac{2m}{\hbar^2} d\epsilon$

$$L^{11} = \int \frac{d^3k}{(2\pi)^3}\, \tau e^2 \left(-\frac{\partial f}{\partial \epsilon}\right) v_x^2$$

$\int d\epsilon \left(-\frac{\partial f}{\partial \epsilon}\right) = 1$

$$= \frac{1}{3}\int \frac{d^3k}{(2\pi)^3}\, \tau e^2 \left(-\frac{\partial f}{\partial \epsilon}\right) v^2$$

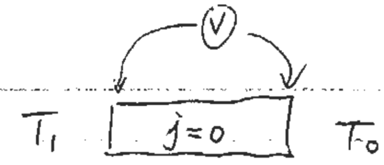
$$= \frac{1}{3}\frac{1}{(2\pi)^3}\, 2\pi k_F \frac{2m}{\hbar^2}\, \tau e^2 v_F^2$$

$$= \frac{1}{3}\frac{1}{(2\pi)^3}\frac{4\pi}{m}\, \tau e^2 k_F^3$$

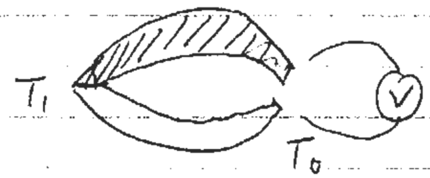
$$\boxed{n = \frac{4}{3}\pi k_F^3 \frac{1}{(2\pi)^3}}$$

$$= \frac{e^2\tau n}{m}$$ (result of Drude model) ✓

Thermopower



$$\vec{\mathcal{E}} = Q \nabla T \qquad (\vec{j}=0)$$



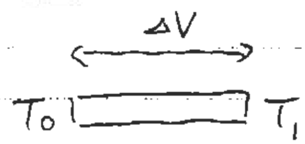
$$Q = \frac{L^{12}}{L^{11}}$$

$$\sim \frac{n^{1/3} e \cancel{\tau} k_B^2 T}{\hbar^2} \Big/ \frac{e^2 \cancel{\tau} n}{m}$$

inter electron spacing

$$= \frac{k_B}{e}\left(\frac{m k_B T}{n^{2/3}\hbar^2}\right) \sim \frac{k_B}{e}\,\frac{1}{(n^{1/3}\lambda_T)^2} \sim \frac{k_B}{e}\left(\frac{\ell}{\lambda_T}\right)^2$$

Let $Q = Q_0 \frac{k_B}{e}$



$$eL\,\mathcal{E} = Q_0\, k_B (T_1 - T_0)$$

$1\,eV \sim 1160\,K$

$$\mathcal{E} = \frac{1}{1160} Q_0 \left(\nabla\cdot T\right)_{\frac{\text{Kelvin}}{\text{meter}}} \left(\frac{\text{Volts}}{\text{meter}}\right)$$

Einstein relation:

One particle in a gas in $V = \frac{1}{2} K x^2$ potential

Boltzman distribution

$$P(x) = \cdot \; e^{-\frac{\beta}{2} K x^2}$$

$$\langle \tfrac{1}{2} K x^2 \rangle = \tfrac{1}{2} k_B T \implies \langle x^2 \rangle = \frac{k_B T}{K}$$

$\langle x^2 \rangle$ from point view of diffussion

When $V = 0$,

Prob. distribution $P(x,t)$ satisfies diffussion equation

$$\boxed{\frac{\partial P}{\partial t} - D \frac{\partial^2 P}{\partial x^2} = 0}$$

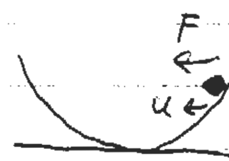
$\langle x^2 \rangle = 2 \underset{\uparrow}{D} t$ (in 1D)

$\uparrow$ diffusion const.

$\boxed{\langle x^2 \rangle = 2 d D t}$ (in d-dimensions)

$= N_t \lambda_0 \leftarrow$ step length.

$\uparrow$ # of steps



When $V \neq 0$, particle experience a force F. F induces a drift $\boxed{u = \eta F}$

$\uparrow$ mobility

drift make $\langle x^2 \rangle$ finite in $t = 0$ limit

We require $\langle x^2 \rangle = \dfrac{k_B T}{K} \implies$ $\boxed{\text{friction } \eta \text{ related to diffusion } D}$

Diffusion with Force.

$$(n_{(x)} = N P_{(x)})$$

$$\vec{j} = n\vec{u} - D\nabla n$$

Diffusion equation: from $\frac{\partial n}{\partial t} + \nabla\cdot\vec{j} = 0$

$$\vec{u} = \eta\vec{F}$$

$$\Rightarrow \boxed{\frac{\partial n}{\partial t} + \eta\nabla\cdot(\vec{F}\,n) - D\nabla^2 n = 0}$$

Steady state (equilibrium state) $\frac{\partial n}{\partial t} = 0$

$$\eta\nabla\cdot(\vec{F}n) - D\nabla^2 n = 0$$

or $\eta\vec{F}n = D\nabla n$.

if $\vec{F} = -\nabla V \Rightarrow \quad D\nabla n = -\eta n\nabla V$

$$\nabla \ln n = -\frac{\eta}{D}\nabla V$$

In equilibrium, we should have

$$n \propto e^{-\beta V} \Rightarrow \nabla \ln n = -\beta\nabla V$$

$$\Rightarrow \frac{\eta}{D} = \beta \quad \text{or} \quad \boxed{D = \eta k_B T}$$ Einstein relation

fluctuation $\Leftrightarrow$ dissipation (diffusion $\Leftrightarrow$ friction)

D $\quad\quad$ η

$\boxed{\langle x^2\rangle = 2dDt}$ $\quad$ $\boxed{D = \eta k_B T}$ $\quad$ $\boxed{\frac{\partial n}{\partial t} - D\nabla^2 n = 0}$ $\quad$ $\boxed{\vec{u} = \eta\vec{F}}$

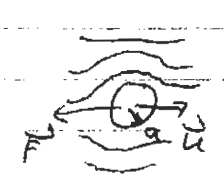
Random walk of particle in Water

viscosity of H_2O at $20\,C^\circ$

$\nu = 0.01$ Poise (dyne, sec/cm²)

$$\nu = \text{Pressure} \Big/ \frac{du}{dx} = \text{dyne}/\text{cm}^2 \Big/ \frac{\text{cm}}{\text{sec}}\Big/\text{cm} = (\text{dyne}/\text{cm}^2)\cdot \text{sec}$$

$\vec{F} = ?$



$$F = p \cdot a^2$$
$$= a^2 \cdot \nu \cdot \frac{du}{dx} = a^2 \nu \frac{u}{a} = a \nu u$$

$\boxed{\text{valid only when } a \gg \lambda}$

$$\vec{F} = \underbrace{6\pi}_{18}\, a \nu \vec{u} \quad \Rightarrow \quad \eta = (6\pi a \nu)^{-1}$$

Einstein's relation

$$D = k_B T \eta$$

$$\langle \vec{x}^2 \rangle = 6 D t = 6 k_B T \eta t$$
$$= \frac{k_B T t}{\pi a \nu}$$

$$k_B = \frac{\langle \vec{x}^2 \rangle \pi a \nu}{T\ t}$$

$R = k_B N_A$

Avogadro's number

$$\sqrt{\langle \vec{x}^2 \rangle} = 8.9\,\mu m \sqrt{\frac{t}{1\,\text{min}} \Big/ \left(\frac{a}{1\,\mu m}\right)} \quad \text{in water}$$

For gas $\quad \nu \sim \dfrac{\bar{p}}{\sigma}$ ← average momentum, ← scattering crossection $\qquad \nu = \dfrac{\sqrt{12 m k_B T}}{3\sigma}$

$$\langle x^2 \rangle \approx \frac{k_B T t}{\pi a\ \bar{p}/\sigma} = \frac{\sigma k_B T}{a \bar{p}} t$$
$$= \frac{\sigma \bar{v}}{a} t = \sigma \frac{\bar{v} t}{a}$$

Langevin equation — Dynamics in random walk

$$\frac{dv}{dt} = \frac{F_{ex}}{m} + \frac{F'}{m}$$

collision force

but $\frac{d\langle v\rangle}{dt} = \frac{F_{ex}}{m}$ if $\langle F'\rangle = 0$

no friction $\Rightarrow$ $\langle F'\rangle = -\gamma v$

damping coeff

$$\boxed{\frac{dv}{dt} + \gamma v = \frac{F_{ex}}{m} + \frac{\tilde{F}}{m} \qquad \langle \tilde{F}\rangle = 0}$$

steady state $\langle v\rangle = \frac{1}{m\gamma} F_{ex}$

describe the true motion of Brownian particle.

$\boxed{\frac{1}{m\gamma} = \eta}$ mobility

$\langle x^2\rangle$:

$$x\frac{d^2x}{dt^2} + \gamma x\frac{dx}{dt} = \frac{\tilde{F}}{m}x \qquad (\text{Let } F_{ex} = 0)$$

$$\langle x\frac{d^2x}{dt^2} + \gamma x\frac{dx}{dt}\rangle = 0 \qquad \boxed{\langle \tilde{F}x\rangle = 0}$$

$$\frac{d}{dt}x\frac{dx}{dt} - \left(\frac{dx}{dt}\right)^2 = \frac{1}{2}\frac{d^2}{dt^2}x^2 - \left(\frac{dx}{dt}\right)^2$$

$$\frac{1}{2}\frac{d^2}{dt^2}\langle x^2\rangle + \frac{1}{2}\gamma\frac{d\langle x^2\rangle}{dt} - \langle\left(\frac{dx}{dt}\right)^2\rangle = 0$$

$$\frac{k_B T/2}{m/2}$$

Let $Y = \langle x^2\rangle$

$$\therefore \frac{d^2}{dt^2}Y = -\gamma\frac{dY}{dt} + 2\frac{k_B T}{m}$$

"mass" = 1 "friction" "Force"

$$\boxed{D = \frac{k_B T}{m\gamma} = k_B T\eta}$$

$t \to \infty$ $\gamma\frac{dY}{dt} = 2\frac{k_B T}{m}$

$$Y = \langle x^2\rangle = \frac{2k_B T}{m\gamma}t = 2Dt$$

In general

$$\langle x^2\rangle = A + B e^{-\gamma t} + \frac{2k_BT}{m\gamma} t$$

at $t=0$ $\quad \langle x\rangle = 0 \qquad \langle x^2\rangle = \cancel{x}\, t^2$

$$\frac{d\langle x^2\rangle}{dt} = \langle x^2\rangle = 0 \Big|_{t=0}$$

$$A + B = 0 \qquad A - \gamma B + \frac{2k_BT}{m\gamma} = 0$$

$$\boxed{\langle x^2\rangle = \frac{2k_BT}{m\gamma}\left[t - \gamma^{-1}(1-e^{-\gamma t})\right]}$$

$$= 2k_BT\eta\,\Big[\underset{\substack{\uparrow\\ \text{random}\\ \text{walk}}}{t} - m\eta\underbrace{(1-e^{-t/m\eta})}_{\tau\ \text{crossover}}\Big]$$

$t \Rightarrow \infty$: $\quad \langle x^2\rangle = 2k_BT\eta\, t - m\eta$

$t \Rightarrow 0$: $\quad \langle x^2\rangle = 2k_BT\eta \cdot \frac{1}{2}\frac{1}{(m\eta)}\, t^2$

$$= \frac{k_BT}{m} t^2 \qquad \frac{1}{2}mv^2 = \frac{1}{2}k_BT \quad !!$$

$\langle v^2\rangle$:

$$v\frac{dv}{dt} + \gamma v^2 = \frac{v\tilde{F}}{m}$$

$$\frac{1}{2}\frac{d}{dt}\langle v^2\rangle + \gamma\langle v^2\rangle = \frac{\langle v\tilde{F}\rangle}{m} \qquad \langle v\tilde{F}\rangle \neq 0$$

$\hat{L}$ generated by $\tilde{F}$

from $\frac{dv}{dt} + \gamma v = \frac{\tilde{F}}{m} \Rightarrow \boxed{v = v_0 e^{-\gamma t} + \int_{-\infty}^{t} e^{-\gamma(t-t')}\frac{\tilde{F}(t')}{m}dt'}$

$$\frac{dv}{dt} = -\gamma v_0 e^{-\gamma t} + \frac{\tilde{F}(t)}{m} + \int_{-\infty}^{t} -\gamma\, e^{-\gamma(t-t')}\frac{\tilde{F}(t')}{m}dt'$$

$$\langle v\tilde{F}\rangle = \frac{1}{m}\int_{-\infty}^{t}\langle\tilde{F}(t)\tilde{F}(t')\rangle\, e^{-\gamma(t-t')}\,dt'$$

$$= \frac{1}{m}\int_{0}^{\infty} K(s)\, e^{-\gamma s}\, ds \qquad \boxed{K(s) = \langle F(t+s)F(t)\rangle}$$

$\tau(t-t')$

$$\frac{1}{2}\frac{d}{dt}\langle v^2\rangle + \gamma\langle v^2\rangle = \frac{1}{m^2}K_\gamma$$

$$K_\gamma = \int_0^\infty K(s)e^{-\gamma s}\,ds$$

$$\langle v^2\rangle = A\,e^{-2\gamma t} + \frac{1}{m^2\gamma}K_\gamma$$

at $t \to \infty$ $\quad \langle v^2\rangle = \frac{1}{m^2\gamma}K_\gamma$ $\qquad$ random force generate fluctuations of v

$$= \frac{k_B T}{m}$$

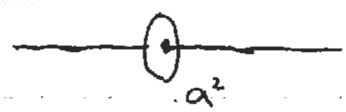
$$\Rightarrow \boxed{m\gamma = \frac{1}{k_B T}K_\gamma}$$

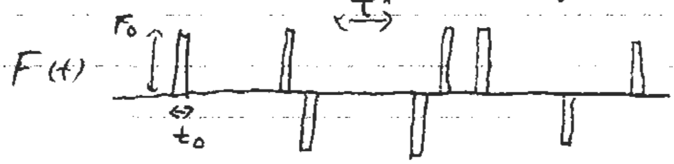
Correlation of random force determine the friction

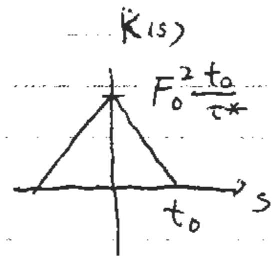
$$\boxed{\frac{1}{\eta} = \frac{1}{k_B T}\int_0^\infty \langle F(s)F(0)\rangle e^{-t/\eta m}\,ds}$$

↑ friction coeff. $\quad F = \frac{1}{\eta}v$

Friction of a disk in gas







$$\langle F(s)F(0)\rangle = 0 \qquad \text{if} \quad s > t_0$$

$$\langle F(0)F(0)\rangle = F_0^2\frac{t_0}{\tau^*}$$

assume $t_0 \ll m\eta = \tau$ relaxation time

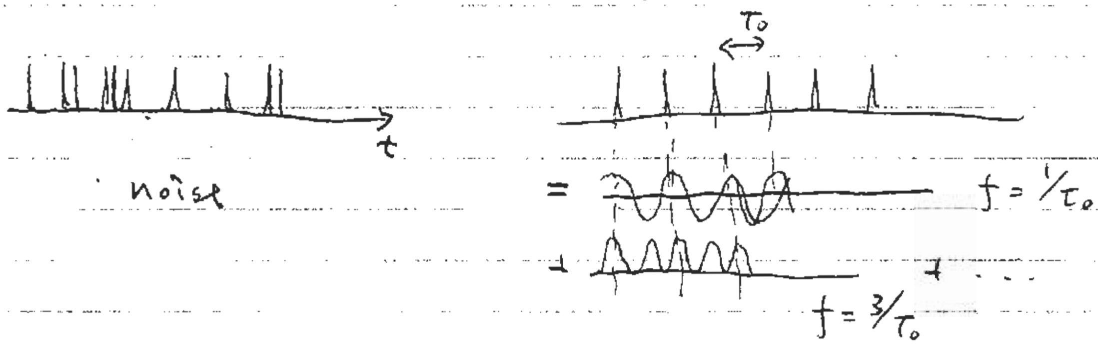
$$\int_0^\infty K(s)\,\cancel{e^{-t/\eta m}}\,ds = \frac{F_0^2 t_0^2}{2\tau^*} = \frac{(2mv_x)^2}{2\tau^*} = \frac{2mk_BT}{\tau^*}$$

$$\frac{1}{\eta} = \frac{2m}{\tau^*} \doteq 2mna^2\bar{v}\Big|_{\text{valid if } a\ll\lambda} \qquad \tau^* \sim 1/na^2\bar{v}$$

hydro-dynamics: $\frac{1}{\eta} = 6\pi\nu a = \pi nm\bar{v}\,a\lambda\Big|_{\text{if } a\gg\lambda}$ $\qquad a^2 \leftrightarrow a\lambda$

$\nu = \frac{1}{6}n\bar{v}m\lambda$

Noise and Noise spectrum



harmonics

Power P & amplitude A

$$P \propto A^2$$

Light $\quad P \propto \vec{E}^2$

$$P = \frac{V^2}{R}$$

Total energy:

$$E = \int_0^{t_\infty} dt\, A^2(t) \qquad A(\omega) = \int dt\, A(t)\, e^{i\omega t}$$

$$A(t) = \int \frac{d\omega}{2\pi} A(\omega)\, e^{-i\omega t}$$

$$= \int \frac{d\omega}{2\pi} A(\omega) A(-\omega)$$

$$= \int_{-\infty}^{+\infty} \frac{d\omega}{2\pi} |A(\omega)|^2 = \int_0^{\infty} \frac{d\omega}{\pi} |A(\omega)|^2$$

Total energy in $[\omega, \omega + d\omega]$ $\quad \frac{d\omega}{2\pi} |A(\omega)|^2$

Power in $[\omega, \omega + d\omega] = \frac{|A(\omega)|^2}{\pi t_\infty} d\omega$

$$\boxed{\frac{|A(\omega)|^2}{\pi t_\infty} = \text{Power spectrum}}$$

How to calculate $\frac{|A(\omega)|^2}{\pi t_\infty}$?

① $A(\omega) = \int dt\, A(t)\, e^{i\omega t}$ but $A(t)$ random

② $$\left\langle \frac{|A(\omega)|^2}{\pi t_\infty} \right\rangle = \frac{1}{\pi t_\infty} \left\langle \int dt\, A(t)\, e^{i\omega t} \int dt'\, A(t')\, e^{-i\omega t'} \right\rangle$$

$$= \frac{1}{\pi t_\infty} \int dt\, dt' \underbrace{\langle A(t)\, A(t') \rangle}_{\equiv G(t-t')} e^{i\omega(t-t')}$$

$$= \frac{1}{\pi t_\infty} \int dt\, d\tilde{t}\;\; G(\tilde{t})\, e^{i\omega\tilde{t}}$$

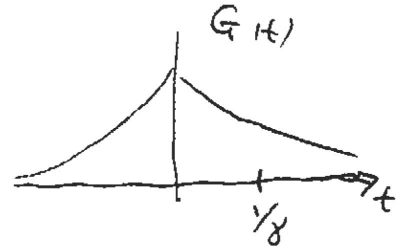
$$= \frac{1}{\pi} \int_{-\infty}^{\infty} d\tilde{t}\, G(\tilde{t})\, e^{i\omega\tilde{t}} = \frac{S(\omega)}{\pi}$$

$$\text{Power spectrum} = \left\langle \frac{|A(\omega)|^2}{\pi t_\infty} \right\rangle = \frac{1}{\pi} \int dt\, G(t)\, e^{i\omega t} = \frac{S(\omega)}{\pi}$$

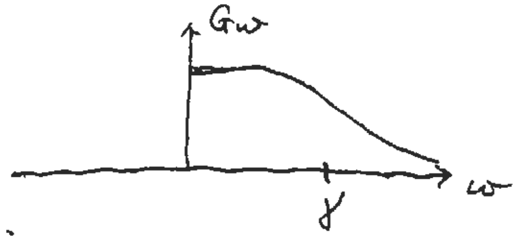
$G(t) = \langle A(t)\, A(0) \rangle$ correlation function

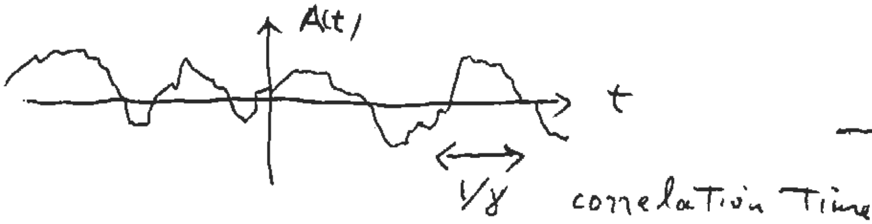
$S(\omega) = \int dt\;\, G(t)\, e^{i\omega t}$

Assume $G(t) = c\, e^{-\gamma|t|}$



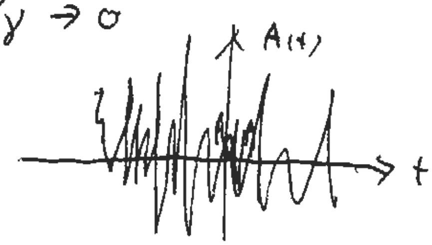
$$S(\omega) = \frac{c\,\gamma/\pi}{\omega^2 + \gamma^2}$$





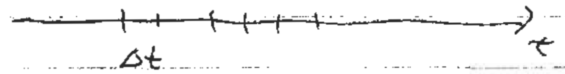
White noise $1/\gamma \to 0$

$S(\omega) = \text{const}.$



$A(t)$ and $A(t+\Delta t)$ have no correlation.

Shot noise



$$I_i = \frac{e}{\Delta t} \, s_i \qquad P(s_i \quad) = p$$

$$P(s_i = 0) = (1-p)$$

$$\langle I_i I_j \rangle = \begin{cases} \left(\frac{pe}{\Delta t}\right)^2 & \text{if } i \neq j \\ \left(\frac{e}{\Delta t}\right)^2 p & \text{if } i = j \end{cases}$$

$$= \begin{cases} I^2 & \text{if } i \neq j \\ \frac{e}{\Delta t} I & \text{if } i = j \end{cases}$$

Note that $\sum_i \left(\langle I_i I_j \rangle - I^2 \right) = \frac{e}{\Delta t} I$

$$\Rightarrow \int dt \, \langle I(t) I(0) \rangle = e I$$

$$\boxed{\langle I(t) I(0) \rangle = I^2 + e I \, \delta(t)} \equiv G(t)$$

$$\text{Power spectrum} = \left\langle \frac{|I(\omega)|^2}{\pi t_{\infty}} \right\rangle$$

$$= \frac{S(\omega)}{\pi}$$

$$S(\omega) = \int dt \, G(t) \, e^{i\omega t} = I^2 (2\pi) \delta(\omega) + e I$$

$$\text{Power spectrum} = 2 I^2 \delta(\omega) + \frac{e I}{\pi}$$

$\boxed{\text{ind. of } \omega \text{ depend on } e\,!}$

<u>Shot noise</u>

Noise spectrum in Brownian motion

Langevin equation:

$$\frac{dv}{dt} + \gamma v = \frac{F(t)}{m} \leftarrow \text{random} \qquad \langle F(t) \rangle = 0$$

In ω-space

$$-i v_\omega + \gamma v_\omega = \frac{F_\omega}{m}$$

$$v_\omega = \frac{F_\omega}{m(\gamma - i\omega)}$$

Now consider

$$\langle v_\omega v_{-\omega} \rangle = \int dt_1 dt_2 \langle v(t_1) v(t_2) \rangle e^{i\omega(t_1 - t_2)}$$

$$= t_\infty \int d\tau \langle v(\tau) v(0) \rangle e^{i\omega\tau}$$

Similarly

$$\langle F_\omega F_{-\omega} \rangle = t_\infty \int d\tau \langle F(\tau) F(0) \rangle e^{i\omega\tau}$$

Noise spectrum

$$\frac{S(\omega)}{\pi} = \frac{1}{\pi} \frac{\langle v_\omega v_{-\omega} \rangle}{t_\infty}$$

$$= \frac{1}{\pi} \frac{1}{m^2(\gamma^2 + \omega^2)} \frac{\langle F_\omega F_{-\omega} \rangle}{t_\infty}$$

$$\boxed{\frac{S(\omega)}{\pi} = \frac{1}{\pi} \frac{1}{m^2(\gamma^2 + \omega^2)} \int d\tau \langle F(\tau) F(0) \rangle e^{i\omega\tau}}$$

Let $\quad K(\omega) = \int d\tau \langle F(\tau) F(0) \rangle e^{i\omega\tau}$

$$\boxed{\frac{S(\omega)}{\pi} = \frac{1}{\pi} \frac{K(\omega)}{m^2(\gamma^2 + \omega^2)}}$$

Let $\langle F(t_1)\, F(t_2) \rangle = 2\pi K \;\delta(t_1 - t_2)$

$$K(\omega) = 2\pi K$$

$$\boxed{\frac{S(\omega)}{\pi} = \frac{2K}{m^2(\gamma^2+\omega^2)}}$$

see page 119

$$m\gamma = \frac{1}{k_B T} K_\gamma \qquad K_\gamma = \int_0^\infty d\tau \,\langle F(\tau) F(0) \rangle\, e^{-\gamma\tau}$$

$$= \frac{1}{k_B T} \pi K \qquad = \pi K_0$$

power spectrum of $v(t)$

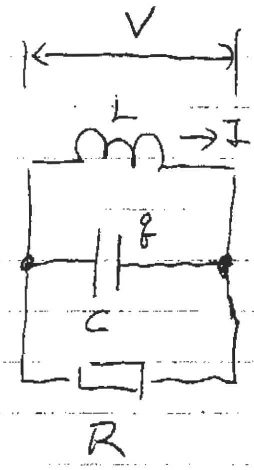
$$P^v_{(\omega)} = \boxed{\frac{S(\omega)}{\pi} = \frac{2k_B T\, m\gamma}{\pi\, m^2(\gamma^2+\omega^2)} = \frac{2k_B T \cdot \gamma}{\pi\, m(\gamma^2+\omega^2)}}$$

$$\langle v_{(t)}^2 \rangle = \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} \frac{\langle v_\omega v_{-\omega} \rangle}{t_\infty} = \int_{-\infty}^{\infty} d\omega \frac{k_B T\, \gamma}{\pi\, m(\gamma^2+\omega^2)} = \frac{k_B T}{m}$$

✓

Power spectrum of velocity is determined by friction γ and mass m (and temperature T)

Noise in RCL circuit



$$V = \frac{q}{C} = L\frac{dI}{dt}$$

$$I = -\frac{dq}{dt} - \frac{V}{R}$$

Let $\quad x = LI$

$$\frac{1}{C}\frac{dq}{dt} = \frac{d^2x}{dt^2} = \frac{1}{C}\left(-I - \frac{V}{R}\right)$$

$$\Rightarrow \quad \boxed{C\frac{d^2x}{dt^2} = -\frac{x}{L} - \frac{1}{R}\frac{dx}{dt}}$$

total energy of the system

$$E = \frac{1}{2}LI^2 + \frac{1}{2}CV^2$$

$$= \frac{1}{2}\frac{1}{L}x^2 + \frac{1}{2}C\dot{x}^2$$

RCL circuit $=$ particle of mass $C (=m)$ in $\frac{1}{2L}x^2$ potential

Friction coeffecient $\quad \gamma m = \frac{1}{R} \quad$ or $\quad \boxed{\gamma = \frac{1}{RC}}$

Let $\quad L = \infty$:

power spectrum of $V\ (=\dot{x})$

$$P^V(\omega) = \frac{2k_BT\frac{1}{RC}}{\pi C\left[\left(\frac{1}{RC}\right)^2 + \omega^2\right]} = \frac{2k_BTR}{\pi(1+\omega^2R^2C^2)}$$

$$P^V(f) = 2\pi P^V(\omega) = 4k_BTR/(1+\omega^2R^2C^2) = 4k_BTR\Big|_{C=0}$$