# 8.07 Lecture 36: December 10, 2012
# POTENTIALS AND FIELDS

## Maxwell's Equations with Sources:

$$\begin{aligned} &\text{(i)} \quad \vec{\nabla} \cdot \vec{E} = \frac{1}{\epsilon_0}\rho && \text{(iii)} \quad \vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t} \ , \\ &\text{(ii)} \quad \vec{\nabla} \cdot \vec{B} = 0 && \text{(iv)} \quad \vec{\nabla} \times \vec{B} = \mu_0 \vec{J} + \frac{1}{c^2}\frac{\partial \vec{E}}{\partial t} \ , \end{aligned} \tag{1}$$

Question: If we are given the sources $\rho(\vec{r}, t)$ and $\vec{J}(\vec{r}, t)$, can we find $\vec{E}$ and $\vec{B}$? If we accept the proposition that all integrals are in principle doable (at least numerically), then the answer is $\boxed{\text{YES}}$.

# Electromagnetic Potentials

If $\vec{B}$ depends on time, then $\vec{\nabla} \times \vec{E} \neq \vec{0}$, so we cannot write $\vec{E} = -\vec{\nabla} V$. BUT: we can still write

$$\boxed{\vec{B} = \vec{\nabla} \times \vec{A}\ .} \tag{2}$$

Then notice that

$$\vec{\nabla} \times \vec{E} = -\frac{\partial}{\partial t}(\vec{\nabla} \times \vec{A}) \quad \Longrightarrow \quad \vec{\nabla} \times \left( \vec{E} + \frac{\partial \vec{A}}{\partial t} \right) = 0\ . \tag{3}$$

so we can write

$$\vec{E} + \frac{\partial \vec{A}}{\partial t} = -\vec{\nabla} V \quad \Longrightarrow \quad \boxed{\vec{E} = -\vec{\nabla} V - \frac{\partial \vec{A}}{\partial t}\ .} \tag{4}$$

With

$$\boxed{\vec{B} = \vec{\nabla} \times \vec{A}\ , \quad \vec{E} = -\vec{\nabla} V - \frac{\partial \vec{A}}{\partial t}\ ,} \tag{5}$$

the source-free Maxwell equations (ii) and (iii),

$$\begin{array}{llll}
\text{(i)} & \vec{\nabla} \cdot \vec{E} = \dfrac{1}{\epsilon_0} \rho & \text{(iii)} & \vec{\nabla} \times \vec{E} = -\dfrac{\partial \vec{B}}{\partial t}\ , \\
\text{(ii)} & \vec{\nabla} \cdot \vec{B} = 0 & \text{(iv)} & \vec{\nabla} \times \vec{B} = \mu_0 \vec{J} + \dfrac{1}{c^2} \dfrac{\partial \vec{E}}{\partial t}\ ,
\end{array}$$

are automatically satisfied. We must therefore deal with the two other Maxwell equations, (i) and (iv).

# Maxwell's Other Equations:

$$\text{(i)} \quad \vec{\nabla} \cdot \vec{E} = \frac{1}{\epsilon_0}\rho \quad \Longrightarrow \quad \nabla^2 V + \frac{\partial}{\partial t}(\vec{\nabla} \cdot \vec{A}) = -\frac{1}{\epsilon_0}\rho \; . \tag{6}$$

$$\text{(iv)} \quad \vec{\nabla} \times \vec{B} = \mu_0 \vec{J} + \frac{1}{c^2}\frac{\partial \vec{E}}{\partial t}$$

$$\Longrightarrow \quad \vec{\nabla} \times (\vec{\nabla} \times \vec{A}) = \mu_0 \vec{J} - \frac{1}{c^2}\vec{\nabla}\left(\frac{\partial V}{\partial t}\right) - \frac{1}{c^2}\frac{\partial^2 \vec{A}}{\partial t^2}$$

$$\Longrightarrow \quad \left(\nabla^2 \vec{A} - \frac{1}{c^2}\frac{\partial^2 \vec{A}}{\partial t^2}\right) - \vec{\nabla}\left(\vec{\nabla} \cdot \vec{A} + \frac{1}{c^2}\frac{\partial V}{\partial t}\right) = -\mu_0 \vec{J} \tag{7}$$

where we used

$$\vec{\nabla} \times (\vec{\nabla} \times \vec{A}) = \vec{\nabla}(\vec{\nabla} \cdot \vec{A}) - \nabla^2 \vec{A} \; . \tag{8}$$

# Gauge Transformations

We have already discussed gauge transformations for statics. But it easily generalizes to the full theory of electrodynamics.

Let $\Lambda(\vec{r},t)$ be an arbitrary scalar function. Then, if we are given $V(\vec{r},t)$ and $\vec{A}(\vec{r},t)$, we can define new potentials by a **gauge transformation:**

$$\boxed{\vec{A}' = \vec{A} + \vec{\nabla}\Lambda \ , \quad V' = V - \frac{\partial \Lambda}{\partial t} \ .} \tag{9}$$

Then

$$\vec{B}' = \vec{\nabla} \times \vec{A}' = \vec{\nabla} \times \vec{A} + \vec{\nabla} \times \vec{\nabla}\Lambda = \vec{\nabla} \times \vec{A} = \vec{B} \ . \tag{10}$$

$$\vec{E}' = -\vec{\nabla}V' - \frac{\partial \vec{A}'}{\partial t} = -\vec{\nabla}V - \frac{\partial \vec{A}}{\partial t} + \vec{\nabla}\left(\frac{\partial \Lambda}{\partial t}\right) - \frac{\partial}{\partial t}\vec{\nabla}\Lambda = \vec{E} \ . \tag{11}$$

## Choice of Gauge:

Can use gauge freedom, $\vec{A}' = \vec{A} + \vec{\nabla}\Lambda$, to make $\vec{\nabla} \cdot \vec{A}$ whatever we want.

**Coulomb Gauge:** $$\boxed{\vec{\nabla} \cdot \vec{A} = 0\ .} \tag{12}$$

$$\nabla^2 V + \frac{\partial}{\partial t}(\vec{\nabla} \cdot \vec{A}) = -\frac{1}{\epsilon_0}\rho \quad \Longrightarrow \quad \nabla^2 V = -\frac{1}{\epsilon_0}\rho\ . \tag{13}$$

$V$ is easy to find, but $\vec{A}$ is hard. $V$ responds instantaneously to changes in $\rho$, but $V$ is not measurable. $\vec{E}$ and $\vec{B}$ receive information only at the speed of light.

**Lorentz Gauge:** $$\boxed{\vec{\nabla} \cdot \vec{A} = -\frac{1}{c^2}\frac{\partial V}{\partial t}\ .} \tag{14}$$

$$\Longrightarrow \quad \nabla^2 V - \frac{1}{c^2}\frac{\partial^2 V}{\partial t^2} = -\frac{1}{\epsilon_0}\rho\ . \tag{15}$$

Define

$$\Box^2 \equiv \nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2} = \text{D'Alembertian} \; . \tag{16}$$

Then, in Lorentz gauge,

$$\boxed{\Box^2 V = -\frac{1}{\epsilon_0}\rho \; .} \tag{17}$$

In general, $\vec{A}$ obeys Eq. (7):

$$\left(\nabla^2 \vec{A} - \frac{1}{c^2}\frac{\partial^2 \vec{A}}{\partial t^2}\right) - \vec{\nabla}\left(\vec{\nabla}\cdot\vec{A} + \frac{1}{c^2}\frac{\partial V}{\partial t}\right) = -\mu_0 \vec{J} \; .$$

In Lorentz gauge,

$$\boxed{\Box^2 \vec{A} = -\mu_0 \vec{J} \; .} \tag{18}$$

# Solution to $\Box^2 V = -\frac{1}{\epsilon_0}\rho$

Method: Guess a solution and then show that it works.

We know that

$$\nabla^2 V = -\frac{1}{\epsilon_0}\rho \quad \Longrightarrow \quad V(\vec{r},t) = \frac{1}{4\pi\epsilon_0}\int \mathrm{d}^3x' \, \frac{\rho(\vec{r}\,',t)}{|\vec{r}-\vec{r}\,'|} \ . \tag{19}$$

We try the guess

$$\Box^2 V = -\frac{1}{\epsilon_0}\rho \quad \Longrightarrow \quad \boxed{V(\vec{r},t) = \frac{1}{4\pi\epsilon_0}\int \mathrm{d}^3x' \, \frac{\rho(\vec{r}\,',t_r)}{|\vec{r}-\vec{r}\,'|} \ ,} \tag{20}$$

where

$$\boxed{t_r = t - \frac{|\vec{r}-\vec{r}\,'|}{c} = \text{retarded time.}} \tag{21}$$

Testing the trial solution:

$$\vec{r} = x_i \hat{e}_i \ , \quad \partial_i |\vec{r} - \vec{r}\,'| = \frac{x_i - x_i'}{|\vec{r} - \vec{r}\,'|} \ , \quad t_r = t - \frac{|\vec{r} - \vec{r}\,'|}{c} \ ,$$

$$\partial_i \rho(\vec{r}\,', t_r) = -\frac{1}{c} \dot{\rho}(\vec{r}\,', t_r) \frac{x_i - x_i'}{|\vec{r} - \vec{r}\,'|} \ , \quad \text{where } \dot{\rho} \equiv \frac{\partial \rho(\vec{r}\,', t_r)}{\partial t_r} \ , \tag{22}$$

$$\partial_i \frac{x_i - x_i'}{|\vec{r} - \vec{r}\,'|^3} = 4\pi \delta^3(\vec{r} - \vec{r}\,') \ .$$

Testing the trial solution:

$$\vec{r} = x_i \hat{e}_i \ , \quad \partial_i |\vec{r} - \vec{r}\,'| = \frac{x_i - x_i'}{|\vec{r} - \vec{r}\,'|} \ , \quad t_r = t - \frac{|\vec{r} - \vec{r}\,'|}{c} \ ,$$

$$\partial_i \rho(\vec{r}\,', t_r) = -\frac{1}{c} \dot{\rho}(\vec{r}\,', t_r) \frac{x_i - x_i'}{|\vec{r} - \vec{r}\,'|} \ , \quad \text{where } \dot{\rho} \equiv \frac{\partial \rho(\vec{r}\,', t_r)}{\partial t_r} \ , \tag{22}$$

$$\partial_i \frac{x_i - x_i'}{|\vec{r} - \vec{r}\,'|^3} = 4\pi \delta^3(\vec{r} - \vec{r}\,') \ .$$

Then

$$V(\vec{r}, t) = \frac{1}{4\pi\epsilon_0} \int \mathrm{d}^3 x' \frac{\rho(\vec{r}\,', t_r)}{|\vec{r} - \vec{r}\,'|} \ .$$

Testing the trial solution:

$$\vec{r} = x_i \hat{e}_i \ , \quad \partial_i |\vec{r} - \vec{r}\,'| = \frac{x_i - x_i'}{|\vec{r} - \vec{r}\,'|} \ , \quad t_r = t - \frac{|\vec{r} - \vec{r}\,'|}{c} \ ,$$

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$$\partial_i \frac{x_i - x_i'}{|\vec{r} - \vec{r}\,'|^3} = 4\pi \delta^3(\vec{r} - \vec{r}\,') \ .$$

Then

$$V(\vec{r}, t) = \frac{1}{4\pi\epsilon_0} \int \mathrm{d}^3x' \frac{\rho(\vec{r}\,', t_r)}{|\vec{r} - \vec{r}\,'|} \ .$$

$$\partial_i V = \frac{1}{4\pi\epsilon_0} \int \mathrm{d}^3x' \left[ \frac{-\frac{1}{c}\,\dot{\rho}}{|\vec{r} - \vec{r}\,'|^2} (x_i - x_i') - \frac{\rho}{|\vec{r} - \vec{r}\,'|^3} (x_i - x_i') \right] \ .$$

Testing the trial solution:

$$\vec{r} = x_i \hat{e}_i \ , \quad \partial_i |\vec{r} - \vec{r}\,'| = \frac{x_i - x_i'}{|\vec{r} - \vec{r}\,'|} \ , \quad t_r = t - \frac{|\vec{r} - \vec{r}\,'|}{c} \ ,$$

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$$\partial_i \frac{x_i - x_i'}{|\vec{r} - \vec{r}\,'|^3} = 4\pi \delta^3(\vec{r} - \vec{r}\,') \ .$$

Then

$$V(\vec{r}, t) = \frac{1}{4\pi\epsilon_0} \int d^3x' \frac{\rho(\vec{r}\,', t_r)}{|\vec{r} - \vec{r}\,'|} \ .$$

$$\partial_i V = \frac{1}{4\pi\epsilon_0} \int d^3x' \left[ \frac{-\frac{1}{c}\, \dot{\rho}}{|\vec{r} - \vec{r}\,'|^2} (x_i - x_i') - \frac{\rho}{|\vec{r} - \vec{r}\,'|^3} (x_i - x_i') \right] \ .$$

Testing the trial solution:

$$\vec{r} = x_i \hat{e}_i \ , \quad \partial_i |\vec{r} - \vec{r}\,'| = \frac{x_i - x_i'}{|\vec{r} - \vec{r}\,'|} \ , \quad t_r = t - \frac{|\vec{r} - \vec{r}\,'|}{c} \ ,$$

$$\partial_i \rho(\vec{r}\,', t_r) = -\frac{1}{c}\dot{\rho}(\vec{r}\,', t_r)\frac{x_i - x_i'}{|\vec{r} - \vec{r}\,'|} \ , \quad \text{where } \dot{\rho} \equiv \frac{\partial \rho(\vec{r}\,', t_r)}{\partial t_r} \ , \tag{22}$$

$$\partial_i \frac{x_i - x_i'}{|\vec{r} - \vec{r}\,'|^3} = 4\pi\delta^3(\vec{r} - \vec{r}\,') \ .$$

Then

$$V(\vec{r}, t) = \frac{1}{4\pi\epsilon_0}\int \mathrm{d}^3x'\,\frac{\rho(\vec{r}\,', t_r)}{|\vec{r} - \vec{r}\,'|} \ .$$

$$\partial_i V = \frac{1}{4\pi\epsilon_0}\int \mathrm{d}^3x' \left[\frac{-\frac{1}{c}\,\dot{\rho}}{|\vec{r} - \vec{r}\,'|^2}(x_i - x_i') - \frac{\rho}{|\vec{r} - \vec{r}\,'|^3}(x_i - x_i')\right] \ .$$

$$\partial_i^2 V = \frac{1}{4\pi\epsilon_0}\int \mathrm{d}^3x' \left[-4\pi\rho\delta^3(\vec{r} - \vec{r}\,') + \frac{\frac{1}{c}\,\dot{\rho}}{|\vec{r} - \vec{r}\,'|^2} + \frac{\frac{1}{c^2}\,\ddot{\rho}}{|\vec{r} - \vec{r}\,'|}\right.$$

$$\left. + \frac{\frac{2}{c}\dot{\rho}}{|\vec{r} - \vec{r}\,'|^2} - \frac{\frac{3}{c}\,\dot{\rho}}{|\vec{r} - \vec{r}\,'|^2}\right]$$

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$$\left. + \frac{\frac{2}{c}\dot{\rho}}{|\vec{r}-\vec{r}\,'|^2} - \frac{\frac{3}{c}\,\dot{\rho}}{|\vec{r}-\vec{r}\,'|^2}\right]$$

$$= -\frac{\rho(\vec{r},t)}{\epsilon_0} + \frac{1}{4\pi\epsilon_0 c^2}\int \mathrm{d}^3x' \frac{\frac{\partial^2\rho(\vec{r}\,',t_r)}{\partial t_r^2}}{|\vec{r}-\vec{r}\,'|} \qquad \left(t_r = t - \frac{|\vec{r}-\vec{r}\,'|}{c}\right)$$

$$= -\frac{\rho(\vec{r},t)}{\epsilon_0} + \frac{1}{4\pi\epsilon_0 c^2}\frac{\partial^2}{\partial t^2}\int \mathrm{d}^3x' \frac{\rho(\vec{r}\,',t_r)}{|\vec{r}-\vec{r}\,'|}$$

$$= -\frac{1}{\epsilon_0}\rho + \frac{1}{c^2}\frac{\partial^2 V}{\partial t^2}$$

$$\partial_i^2 V = \frac{1}{4\pi\epsilon_0}\int d^3x' \left[ -4\pi\rho\delta^3(\vec{r}-\vec{r}\,') + \frac{\frac{1}{c}\,\dot{\rho}}{|\vec{r}-\vec{r}\,'|^2} + \frac{\frac{1}{c^2}\,\ddot{\rho}}{|\vec{r}-\vec{r}\,'|} \right.$$

$$\left. + \frac{\frac{2}{c}\dot{\rho}}{|\vec{r}-\vec{r}\,'|^2} - \frac{\frac{3}{c}\,\dot{\rho}}{|\vec{r}-\vec{r}\,'|^2} \right]$$

$$= -\frac{\rho(\vec{r},t)}{\epsilon_0} + \frac{1}{4\pi\epsilon_0 c^2}\int d^3x' \frac{\frac{\partial^2\rho(\vec{r}\,',t_r)}{\partial t_r^2}}{|\vec{r}-\vec{r}\,'|} \qquad \left(t_r = t - \frac{|\vec{r}-\vec{r}\,'|}{c}\right)$$

$$= -\frac{\rho(\vec{r},t)}{\epsilon_0} + \frac{1}{4\pi\epsilon_0 c^2}\frac{\partial^2}{\partial t^2}\int d^3x' \frac{\rho(\vec{r}\,',t_r)}{|\vec{r}-\vec{r}\,'|}$$

$$= -\frac{1}{\epsilon_0}\rho + \frac{1}{c^2}\frac{\partial^2 V}{\partial t^2}$$

***YES!***

# Retarded Time Solutions

$$V(\vec{r},t)=\frac{1}{4\pi\epsilon_0}\int d^3x'\,\frac{\rho(\vec{r}\,',t_r)}{|\vec{r}-\vec{r}\,'|}$$
$$\vec{A}(\vec{r},t)=\frac{1}{4\pi\epsilon_0}\int d^3x'\,\frac{\vec{J}(\vec{r}\,',t_r)}{|\vec{r}-\vec{r}\,'|}\ , \qquad (23)$$

where

$$t_r=t-\frac{|\vec{r}-\vec{r}\,'|}{c}\ . \qquad (24)$$

# Advanced Time Solutions??

$$\boxed{\begin{aligned} V(\vec{r},t) &= \frac{1}{4\pi\epsilon_0}\int d^3x' \, \frac{\rho(\vec{r}\,',t_a)}{|\vec{r}-\vec{r}\,'|} \\ \vec{A}(\vec{r},t) &= \frac{1}{4\pi\epsilon_0}\int d^3x' \, \frac{\vec{J}(\vec{r}\,',t_a)}{|\vec{r}-\vec{r}\,'|} \ , \end{aligned}} \tag{25}$$

where

$$\boxed{t_a = t + \frac{|\vec{r}-\vec{r}\,'|}{c} \ .} \tag{26}$$

Maxwell's equations, and the laws of physics as we know them, make no distinction between the future and the past.

# Mystery of the Arrow of Time

Real Events:

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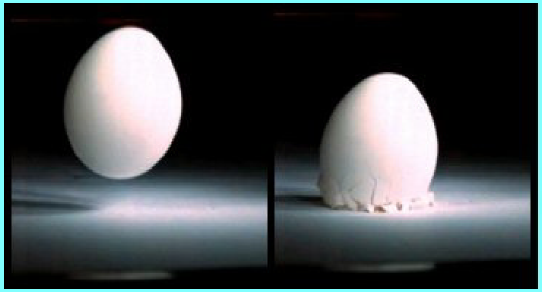
Real Events:



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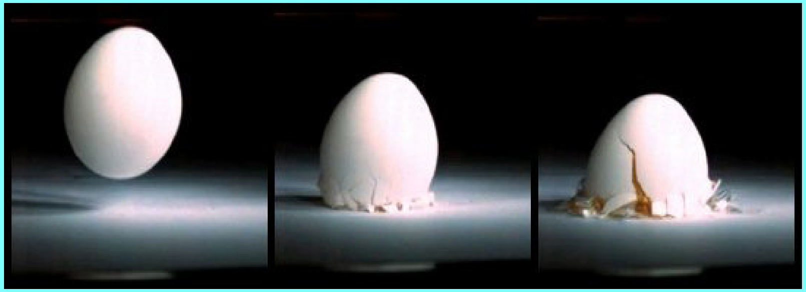
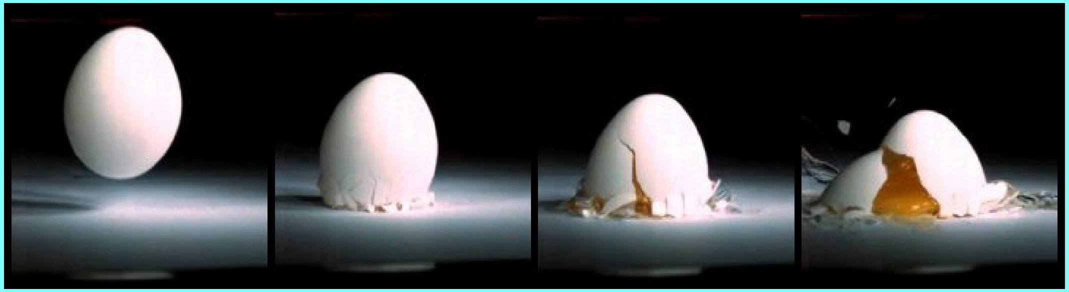
Real Events:



# Mystery of the Arrow of Time

Real Events:



# Mystery of the Arrow of Time

Real Events:

# Mystery of the Arrow of Time

Real Events:



*Arrow of time*

# Mystery of the Arrow of Time

Real Events:



*Arrow of time*

Laws of Physics:

# Mystery of the Arrow of Time

Real Events:



*Arrow of time*

Laws of Physics:

*Time symmetric*

Alan Guth
Massachusetts Institute of Technology

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The universe started out in a low entropy (i.e. highly ordered) state, so the entropy (disorder) has been increasing ever since.

Why did the universe start in a low entropy state?

Who knows?

My preferred explanation:

It is possible that there is no upper limit to the entropy of the universe, so any state in which it may have started is low-entropy compared to what it can be.

# The Fields of a Point Charge

From Eq. (23),

$$V(\vec{r},t) = \frac{1}{4\pi\epsilon_0} \int d^3x' \, \frac{\rho(\vec{r}\,', t_r)}{|\vec{r} - \vec{r}\,'|} \ .$$

For a point charge $q$ moving on a trajectory $\vec{r}_p(t)$,

$$\rho(\vec{r},t) = q\delta^3\Big(\vec{r} - \vec{r}_p(t)\Big) \ , \tag{27}$$

so

$$\begin{aligned} V(\vec{r},t) &= \frac{q}{4\pi\epsilon_0} \int d^3x' \, \frac{\delta^3\Big(\vec{r}\,' - \vec{r}_p(t_r)\Big)}{|\vec{r} - \vec{r}\,'|} \\ &= \frac{q}{4\pi\epsilon_0 |\vec{r} - \vec{r}_p(t_r)|} \int d^3x' \, \delta^3\Big(\vec{r}\,' - \vec{r}_p(t_r)\Big) \ . \end{aligned} \tag{28}$$

But, perhaps surprisingly,

$$Z \equiv \int \mathrm{d}^3x'\, \delta^3\Big(\vec{r}\,' - \vec{r}_p(t_r)\Big) \neq 1 \ , \tag{29}$$

where I am calling the integral $Z$ for future reference. Remember,

$$\delta\Big(g(x)\Big) = \sum_i \frac{\delta(x - x_i)}{|g'(x_i)|} \ , \quad \text{where} \quad g(x_i) = 0 \ , \tag{30}$$

and

$$t_r = t - \frac{|\vec{r} - \vec{r}\,'|}{c} \ .$$

To make things simple, suppose that the particle velocity at $t_r$ points in the $x$-direction. Then

$$\begin{aligned} Z &= \int \mathrm{d}^3x' \delta\Big(x' - x_p(t_r)\Big)\delta\Big(y' - y_p(t_r)\Big)\delta\Big(z' - z_p(t_r)\Big) \\ &= \int \mathrm{d}x'\, \delta\Big(x' - x_p(t_r)\Big) \ , \end{aligned} \tag{31}$$

where the integrals over $y'$ and $z'$ were simple, since $y_p'(t_r) = z_p'(t_r) = 0$.

So we need to evaluate

$$Z = \int \mathrm{d}x'\, \delta\Big(x' - x_p(t_r)\Big) \ , \quad \text{where } t_r = t - \frac{|\vec{r} - \vec{r}\,'|}{c} \ . \tag{32}$$

So, to use our formula,

$$g(x') = x' - x_p\left(t - \frac{|\vec{r} - \vec{r}\,'|}{c}\right) \ , \tag{33}$$

and then

$$g'(x') = 1 - \frac{\mathrm{d}x_p}{\mathrm{d}t}\frac{\mathrm{d}t_r}{\mathrm{d}x'} = 1 + \frac{1}{c}\frac{\mathrm{d}x_p}{\mathrm{d}t}\frac{\mathrm{d}}{\mathrm{d}x'}|\vec{r} - \vec{r}\,'| = 1 + \frac{1}{c}\frac{\mathrm{d}x_p}{\mathrm{d}t}\frac{x' - x}{|\vec{r} - \vec{r}\,'|} \ , \tag{34}$$

and $Z = 1/g'(x')$. Generalizing,

$$Z = \left(1 - \frac{\vec{v}}{c} \cdot \frac{\vec{r} - \vec{r}\,'}{|\vec{r} - \vec{r}\,'|}\right)^{-1} \ . \tag{35}$$

# The Liénard-Wiechert Potentials

Finally,

$$\boxed{V(\vec{r},t) = \frac{1}{4\pi\epsilon_0}\frac{q}{|\vec{r}-\vec{r}_p|\left(1-\frac{\vec{v}_p}{c}\cdot\frac{\vec{r}-\vec{r}_p}{|\vec{r}-\vec{r}_p|}\right)}\ ,} \tag{36}$$

where $\vec{r}_p$ and $\vec{v}_p$ are the position and velocity of the particle at $t_r$. Similarly, starting with

$$\vec{J}(\vec{r},t) = q\vec{v}\delta^3\left(\vec{r}-\vec{r}_p(t)\right) \tag{37}$$

for a point particle, we find

$$\boxed{\vec{A}(\vec{r},t) = \frac{\mu_0}{4\pi}\frac{q\vec{v}_p}{|\vec{r}-\vec{r}_p|\left(1-\frac{\vec{v}_p}{c}\cdot\frac{\vec{r}-\vec{r}_p}{|\vec{r}-\vec{r}_p|}\right)} = \frac{\vec{v}_p}{c^2}V(\vec{r},t)\ .} \tag{38}$$

# The Fields of a Point Charge

Differentiating the Liénard-Wiechert potentials, after several pages, one finds

$$\boxed{\vec{E}(\vec{r},t) = \frac{q}{4\pi\epsilon_0}\frac{|\vec{r}-\vec{r}_p|}{(\vec{u}\cdot(\vec{r}-\vec{r}_p))^3}\left[(c^2-v_p^2)\vec{u} + (\vec{r}-\vec{r}_p)\times(\vec{u}\times\vec{a}_p)\right] \ ,} \tag{39}$$

where

$$\boxed{\vec{u} = c\frac{\vec{r}-\vec{r}_p}{|\vec{r}-\vec{r}_p|} - \vec{v}_p \ .} \tag{40}$$

And

$$\boxed{\vec{B}(\vec{r},t) = \frac{1}{c}\frac{\vec{r}-\vec{r}_p}{|\vec{r}-\vec{r}_p|}\times\vec{E}(\vec{r},t) \ .} \tag{41}$$

Here $\vec{r}_p$, $\vec{v}_p$, and $\vec{a}_p$ are the position, velocity, and acceleration, respectively, of the particle at the retarded time.

If the particle is moving at constant velocity, then the acceleration term in Eq. (39) is absent, and the electric field points along $\vec{u}$. Note that $\vec{u}$ can also be written as

$$\vec{u} = \frac{c}{|\vec{r} - \vec{r}_p|}\left[\vec{r} - \Big(\vec{r}_p + \vec{v}_p(t - t_r)\Big)\right] \; . \tag{42}$$

In this form one can see that, for the case of constant velocity, $\vec{u}$ points outward from the current position of the particle, which is $\vec{r}_p + \vec{v}_p(t - t_r)$.