MASSACHUSETTS INSTITUTE OF TECHNOLOGY
Physics Department

Physics 8.07: Electromagnetism II — October 11, 2012
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PROBLEM SET 5 REVISED*

DUE DATE: Friday, October 12, 2012. Either hand it in at the lecture, or by 6:00 pm in the 8.07 homework boxes.

READING ASSIGNMENT: None. (The material is related to Chapter 3 of Griffiths, which you should have already read.)

CREDIT: This problem set has 90 points of credit plus 10 points extra credit.

PROBLEM 1: A SPHERE WITH OPPOSITELY CHARGED HEMISPHERES *(15 points)*

Griffiths Problem 3.22 (p. 145). (This problem was held over from Problem Set 4.)

PROBLEM 2: A CIRCULAR DISK AT A FIXED POTENTIAL *(20 points)*

This problem is based on Jackson, Problem 3.3 (challenging!).

A thin, flat, conducting circular disk of radius R is located in the x-y plane with its center at the origin, and is mantained at a fixed potential V_0.

(a) With the information that the surface charge density on a disk at fixed potential is proportional to $(R^2 - s^2)^{-1/2}$, where s is the distance out from the center of the disk, show that for $r > R$,

$$V(r, \theta) = \frac{2V_0}{\pi} \frac{R}{r} \sum_{\ell=0}^{\infty} \frac{(-1)^\ell}{2\ell + 1} \left(\frac{R}{r}\right)^{2l} P_{2\ell}(\cos\theta)\,.$$

Hint: First find the field along the z-axis, and then use your knowledge about the general solution to Laplace's equation to infer the angular dependence.

(b) Calculate the capacitance of the disk, defined by $Q = CV$, where Q is the total charge on the disk and V is the potential of the disk relative to $|\vec{r}| \to \infty$.

PROBLEM 3: QUADRUPOLE AND OCTOPOLE TERMS OF THE MULTIPOLE EXPANSION *(20 points)*

Griffiths Problem 3.45 (pp. 158-159). For parts (a) and (d), you can find the answers either by starting with Griffiths' Eq. (3.95) (p. 148) for the multipole expansion, or by starting with the integral expression (Griffiths' Eq. (2.29), p. 84) for the potential and then expanding in powers of $1/r$. Note that Q_{ij} and the octopole moment that you will find are examples of the traceless symmetric tensor formalism that we have discussed in lecture.

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* A typo was fixed in Problem 4(c), Eq. (4.12). In the original version of October 7, 2012, the factor on the right-hand side was inverted.

PROBLEM 4: SPHERICAL HARMONICS AND TRACELESS SYMMETRIC TENSORS *(20 points)*

In class we discussed the fact that any sum of spherical harmonics for a given index ℓ can be written as

$$F_\ell(\theta,\phi) = C^{(\ell)}_{i_1 i_2 \ldots i_\ell} \hat{n}_{i_1} \hat{n}_{i_2} \ldots \hat{n}_{i_\ell} \ , \tag{4.1}$$

where $C^{(\ell)}_{i_1 i_2 \ldots i_\ell}$ is a traceless symmetric tensor, the indices i_1, i_2, $\ldots i_\ell$ are summed 1 to 3, and

$$\hat{n} = \sin\theta\cos\phi\,\hat{e}_1 + \sin\theta\sin\phi\,\hat{e}_2 + \cos\theta\,\hat{e}_3 \tag{4.2}$$

is a unit vector in the direction of (θ,ϕ). The standard functions $Y_{\ell m}(\theta,\phi)$ can be written in this formalism by finding the corresponding traceless symmetric tensors, which we will call $C^{(\ell,m)}_{i_1 i_2 \ldots i_\ell}$:

$$Y_{\ell m}(\theta,\phi) = C^{(\ell,m)}_{i_1 i_2 \ldots i_\ell} \, \hat{n}_{i_1} \hat{n}_{i_2} \ldots \hat{n}_{i_\ell} \ . \tag{4.3}$$

To construct the $C^{(\ell,m)}_{i_1 i_2 \ldots i_\ell} \hat{n}_{i_1} \hat{n}_{i_2} \ldots \hat{n}_{i_\ell}$ explicitly, we introduce the following basis of unit 3-dimensional vectors:

$$\begin{aligned} \hat{u}^{(1)} &\equiv \hat{u}^+ = \frac{1}{\sqrt{2}}(\hat{e}_x + i\hat{e}_y) \\ \hat{u}^{(2)} &\equiv \hat{u}^- = \frac{1}{\sqrt{2}}(\hat{e}_x - i\hat{e}_y) \\ \hat{u}^{(3)} &\equiv \hat{z} = \hat{e}_z \ . \end{aligned} \tag{4.4}$$

These complex-valued vectors are orthonormal in the sense that

$$\hat{u}^{(i)*} \cdot \hat{u}^{(j)} = \delta_{ij} \ , \tag{4.5}$$

but note that

$$\hat{u}^+ \cdot \hat{u}^+ = \hat{u}^- \cdot \hat{u}^- = 0 \ , \tag{4.6}$$

so we find conveniently that constructions such as $\hat{u}^+_i \hat{u}^+_j$ are traceless as well as symmetric. For $m \geq 0$, $C^{(\ell,m)}_{i_1 i_2 \ldots i_\ell} \hat{n}_{i_1} \hat{n}_{i_2} \ldots \hat{n}_{i_\ell}$ is given by

$$C^{(\ell,m)}_{i_1 i_2 \ldots i_\ell} = d_{\ell m} \left\{ \, \hat{u}^+_{i_1} \ldots \hat{u}^+_{i_m} \hat{z}_{i_{m+1}} \ldots \hat{z}_{i_\ell} \, \right\} \ , \tag{4.7}$$

where $\{ \, xxx \, \}$ means the traceless symmetric part of xxx. The constant of proportionality $d_{\ell m}$ is given by

$$d_{\ell m} = \frac{(-1)^m (2\ell)!}{2^\ell \ell!} \sqrt{\frac{2^m \, (2\ell+1)}{4\pi \, (\ell+m)! \, (\ell-m)!}} \ . \tag{4.8}$$

For $m \leq 0$,

$$C^{(\ell,m)}_{i_1 i_2 \ldots i_\ell} = d_{\ell m} \left\{ \, \hat{u}^-_{i_1} \ldots \hat{u}^-_{i_{|m|}} \hat{z}_{i_{|m|+1}} \ldots \hat{z}_{i_\ell} \, \right\} = C^{(\ell,|m|)*}_{i_1 i_2 \ldots i_\ell} \ . \tag{4.9}$$

(a) Use this formalism to show that

$$Y_{31}(\theta,\phi) = -\frac{1}{4}\sqrt{\frac{21}{4\pi}}\,(5\cos^2\theta - 1)\sin\theta\, e^{i\phi}\ . \tag{4.10}$$

Hint: You might benefit from that fact that Eq. (4.3) can alternatively be written as

$$Y_{\ell m}(\theta,\phi) = C^{(\ell,m)}_{i_1 i_2 \ldots i_\ell}\,\{\,\hat{n}_{i_1}\hat{n}_{i_2}\ldots\hat{n}_{i_\ell}\,\}\ , \tag{4.11}$$

since $\{\,\hat{n}_{i_1}\hat{n}_{i_2}\ldots\hat{n}_{i_\ell}\,\}$ differs from $\hat{n}_{i_1}\hat{n}_{i_2}\ldots\hat{n}_{i_\ell}$ only by terms proportional to kronecker δ-functions, and such terms will give no contribution because $C^{(\ell,m)}_{i_1 i_2 \ldots i_\ell}$ is traceless. Once the expansion of $\{\,\hat{n}_{i_1}\hat{n}_{i_2}\ldots\hat{n}_{i_\ell}\,\}$ is written out explicitly, there is no need to take the traceless symmetric part in the evaluation of $C^{(\ell,m)}_{i_1 i_2 \ldots i_\ell}$ using Eq. (4.7). It is a little easier to write the traceless symmetric part of $\hat{n}_{i_1}\hat{n}_{i_2}\ldots\hat{n}_{i_\ell}$ than it is to find the traceless symmetric part of the right-hand side of Eq. (4.7), since $\hat{n}_{i_1}\hat{n}_{i_2}\ldots\hat{n}_{i_\ell}$ is already symmetric.

(b) Use this formalism to derive a general expression for $Y_{\ell\ell}$ that is valid for all ℓ.

(c) For cases with azimuthal symmetry, it is sufficient to use $C^{(\ell)}_{i_1 i_2 \ldots i_\ell}$ tensors that are constructed as traceless symmetric parts of products of $\hat{z}$, which corresponds to what in the usual formalism are called Legendre polynomials. The precise connection is that

$$\begin{aligned} P_\ell(\cos\theta) &= \frac{(2\ell)!}{2^\ell(\ell!)^2}\{\,\hat{z}_{i_1}\hat{z}_{i_2}\ldots\hat{z}_{i_\ell}\,\}\,\hat{n}_{i_1}\hat{n}_{i_2}\ldots\hat{n}_{i_\ell} \\ &= \frac{(2\ell)!}{2^\ell(\ell!)^2}\,\hat{z}_{i_1}\hat{z}_{i_2}\ldots\hat{z}_{i_\ell}\,\{\,\hat{n}_{i_1}\hat{n}_{i_2}\ldots\hat{n}_{i_\ell}\,\}\ . \end{aligned} \tag{4.12}$$

Use this relation to evaluate $P_4(\cos\theta)$, and compare your result with Table 3.1 of Griffiths (p. 138).

PROBLEM 5: THE PRECISE ELECTRIC FIELD OF AN ELECTRIC DIPOLE *(15 points plus 10 points extra credit)*

Griffiths Problem 3.42 (p. 157, *15 points*). Griffiths felt (and I agree) that the answers to this problem are sufficiently subtle that he should give the answers in the statement of the problem. Your job, then, is to *explain* these answers.

(c) *(10 points extra credit)* Note that Griffiths found the δ-function term in the expression for the electric field of a dipole by starting with an ill-defined expression (i.e., $V_{\text{dip}}(r,\theta) = p\cos\theta/(4\pi\epsilon_0 r^2)$ is not differentiable at $r = 0$), then calculating a wrong answer, and then noticing that the answer violated a general theorem. He then asked what could be changed in the answer to avoid claiming a result which is demonstrably false. He got the right answer, but one might hope to find a way to make everything well-defined from the start. The method of distributions, or generalized functions, is exactly that.

To define $V_{\rm dip}$ as a distribution, we define it in terms of what happens when we multiply it by a test function $\varphi(\vec{r})$ and then integrate over all space:

$$F_{V_{\rm dip}}[\varphi(\vec{r})] \equiv \int \varphi(\vec{r}) \left(\frac{p\cos\theta}{4\pi\epsilon_0 r^2} \right) \mathrm{d}^3x \ . \tag{5.1}$$

The test function is required to be smooth, and to fall off rapidly at large $|\vec{r}|$. Note that the integral is well-defined in spite of the factor of $1/r^2$, since the singularity is canceled by the measure $\mathrm{d}^3x = r^2\,\mathrm{d}r\,\sin\theta\,\mathrm{d}\theta\,\mathrm{d}\phi$. The derivative $\partial_i V_{\rm dip} \equiv \partial V_{\rm dip}/\partial x^i$ is defined in the sense of distributions by formally integrating by parts. That is, one might naively differentiate by writing

$$F_{\partial_i V_{\rm dip}}[\varphi(\vec{r})] \underset{\text{naively}}{=} \int \varphi(\vec{r})\partial_i \left(\frac{p\cos\theta}{4\pi\epsilon_0 r^2} \right) \mathrm{d}^3x \ , \tag{5.2}$$

but this integral is ill-defined. The real definition in the sense of distributions is given by

$$F_{\partial_i V_{\rm dip}}[\varphi(\vec{r})] = -\int \partial_i\varphi(\vec{r}) \left(\frac{p\cos\theta}{4\pi\epsilon_0 r^2} \right) \mathrm{d}^3x \ . \tag{5.3}$$

By definition there is no surface term associated with the integration by parts. One can intuitively justify this by the assumption that $\varphi(\vec{r})$ falls of rapidly with $|\vec{r}|$, but in the logic of distribution theory, Eq. (5.3) is simply the definition of the derivative of the distribution $V_{\rm dip}$.

The integral on the right-hand side of Eq. (5.3) is a well-defined, ordinary integral. The goal is to show that this gives Griffith's expression (Eq. (3.106, p. 157)) for the electric field of a dipole, δ-function term included. One way to do this is to separate the integral into two pieces, $r < \epsilon$ and $r > \epsilon$, where ϵ is a positive number which will in the end be taken to zero. Then

$$F_{\partial_i V_{\rm dip}}[\varphi(\vec{r})] = F^{(1)}_{\partial_i V_{\rm dip}}[\varphi(\vec{r})] + F^{(2)}_{\partial_i V_{\rm dip}}[\varphi(\vec{r})] \ , \tag{5.4}$$

where

$$\begin{aligned} F^{(1)}_{\partial_i V_{\rm dip}}[\varphi(\vec{r})] &= -\int_{r>\epsilon} \partial_i\varphi(\vec{r}) \left(\frac{p\cos\theta}{4\pi\epsilon_0 r^2} \right) \mathrm{d}^3x \\ F^{(2)}_{\partial_i V_{\rm dip}}[\varphi(\vec{r})] &= -\int_{r<\epsilon} \partial_i\varphi(\vec{r}) \left(\frac{p\cos\theta}{4\pi\epsilon_0 r^2} \right) \mathrm{d}^3x \ . \end{aligned} \tag{5.5}$$

Looking at $F^{(2)}_{\partial_i V_{\rm dip}}[\varphi(\vec{r})]$, one can see that it vanishes in the limit $\epsilon \to 0$. To see this, write

$$\left| F^{(2)}_{\partial_i V_{\rm dip}}[\varphi(\vec{r})] \right| \leq \frac{|p|}{4\pi\epsilon_0} \int_0^\pi \int_0^{2\pi} \int_0^\epsilon \max_{r<\epsilon}\,(\partial_i\varphi)\ \mathrm{d}r\,\mathrm{d}\phi\,\mathrm{d}\theta \leq \text{const}\ \epsilon \ . \tag{5.6}$$

Here $\max_{r<\epsilon}(\partial_i\varphi)$ represents the maximum value of $|(\partial_i\varphi)|$ in the region $r<\epsilon$, which is guaranteed by the smoothness of $\varphi(\vec{r})$ to approach a constant as $\epsilon \to 0$. We also used the facts that $|\sin\theta| \leq 1$ and $|\cos\theta| \leq 1$. Since we will evaluate Eq. (5.4) in the limit $\epsilon \to 0$, we can forget about $F^{(2)}_{\partial_i V_{\rm dip}}[\varphi(\vec{r})]$.

Since the integral in $F^{(1)}_{\partial_i V_{\rm dip}}[\varphi(\vec{r})]$ has a lower limit >0, we can evaluate it by integrating by parts. We get no surface term at $|\vec{r}|=\infty$, by the properties of φ, but we do find a surface term at $r=\epsilon$:

$$F^{(1)}_{\partial_i V_{\rm dip}}[\varphi(\vec{r})] = -\int_{r>\epsilon} \partial_i \left[\varphi(\vec{r})\left(\frac{p\cos\theta}{4\pi\epsilon_0 r^2}\right)\right] \mathrm{d}^3x + \int_{r>\epsilon} \varphi(\vec{r})\partial_i\left(\frac{p\cos\theta}{4\pi\epsilon_0 r^2}\right) \mathrm{d}^3x \ . \tag{5.7}$$

AT LAST: THE HOMEWORK PROBLEM:

Evaluate the right-hand side of Eq. (5.7), and show that it is equivalent to

$$F^{(1)}_{\partial_i V_{\rm dip}}[\varphi(\vec{r})] = \int \varphi(\vec{r})\left\{\frac{1}{4\pi\epsilon_0}\frac{1}{r^3}\left[3(\vec{p}\cdot\hat{r})\hat{r}_i - p_i\right] - \frac{1}{3\epsilon_0}p_i\delta^3(\vec{r})\right\} \mathrm{d}^3x \ . \tag{5.8}$$

One further hint: The identity proven in Problem 7(a) of Problem Set 1 may prove useful.