PROFESSOR: Today, we continue with scattering. And we begin by reviewing what were the main ideas that have already been explored. And here they are. I've summarized the main results on the blackboard.

And we begin with an expansion of our solutions for the case of central potentials. This psi effect represents a solution away from the scattering center. The scattering center is this region where there is a potential that the particles fill. And this solution that we've written on top is a solution written for spherically symmetric potentials in the case that the potential is 0. That is, that solution is valid away from the potential.

We constructed it with the free particle solutions-- the jl and the nl of kr-- of the spherical Schrodinger or radial Schrodinger equation. In particular, we identified these objects called phase shifts that we discussed in the process of scattering with spherical waves. And the result is that the way we introduce the phase shifts, the tangent of the phase shift is determined by the ratio of these two coefficients, Bl over Al. So there's a phase shift for every value of l beginning from l equals 0.

Then we have that given that relation, the solution above, when you use asymptotic expansions for jl and nl, takes that form with a sine of kr minus l pi over 2 plus a phase shift, which is another way where you can read the phase shift of your solution. We're going to focus today for a little while in understanding the physics of the phase shifts and how you find the phase shifts.

But supposing you find the phase shifts and that was the derivation that was very non-trivial that we did last time, we find what this scattering amplitude is in terms of the phase shift. So this was our main hard work establishing that formula.

Once you have that formula, that f of k in preliminary work was shown to represent the scattering solution far away from the source. Far away from the source we have the incoming wave and the outgoing spherical wave modulated by fk of theta. Now, in general, you can have an fk of theta and phi, but then the phase shift method is not quite suitable for that. For that, we will discuss some other methods today that can allow us to do that.

So once you have this f of k, you know the wave far away. You know how it contributes to the cross-section, and you know how the phase shifts. Each phase shift contributes to a total cross-section, which is given here. So these were our main results.

So let's leave this results here and discuss your general computation of a phase shift. You're given a problem, maybe like the problems in the p set, and you need to find the phase shift. So where do you begin? There's nothing in there that seems to say, OK, this is the way you're going to find the phase shit.

So let's do this general computation of the phase shift. Computation of the phase shifts. So I think you all know-- and there's clear intuition-- that if you want to figure out the phase shift, at some point you have to look at this region, r less than a, where the potential is. Because after all, the phase shift depends on the potential. So that's unavoidable. You have to get dirty with that potential and solve it.

So what is the equation that you have to solve? It's the radial Schrodinger equation. So you have to solve-- you set energy in the Schrodinger equation equal h squared k squared over 2m because we're using k from the beginning to represent the energy of the state. So energy is going to be that. And the radial Schrodinger equation is minus h squared over 2m, d second dr squared plus the potential.

Finally, the potential shows up. You have to solve that. Plus the potential centrifugal barrier, h squared l times l plus 1 over 2m r squared. All that acting on a solution u sub l-- I will put the k as well of r-- equal the energy, which is h squared k squared over 2m ulk of r.

So this step cannot be avoided. You have to solve it. You have to solve this equation. So you solve this for r less than a. This for r less than a. You have some hints of what you're going to encounter because you know from solving radial equations that ul behaves like r to the l plus 1 as r goes to 0. That's a boundary condition for the l-th wave.

And well, with that, you're supposed to solve for this ul. OK. So you solve for this ul. And so what? Where are the phase shifts? You're done with it. And you need to know where are the phase shifts. Well, remember our notation is that we have a radial function, l of k of r.

In the radial equation you know that the radial part of the solution of the Schrodinger equation is-- the full solution is a radial part. So ylm. In this case, yl0. But this radial thing is ulk of r over r. That is presumably something you still remember, that the radial solution was u over r.

OK. So you've solved it. You've got this quantity. And you say, all right, what do I do next? Well, what you have to do next is to match to the solutions that lie outside. There is these solutions that hold for r greater than a, where the potential is 0. So those have to be matched to this solution.

So let me draw a little diagram. We have diagram for r. And here is a. In this region, the solution is Rlk of r. That's the solution. And this solution here, this is for the l-th partial wave. The solution outside is the l-th partial wave that I've written there.

So the solution here is some Al Jl of kr plus Bl nl of kr. So this one you've determined already. You did solve the Schrodinger equation. We cannot help you. Now it depends on what potential you have. You have to solve it each time. But you now have the solution up to A. But you know the general form of the solution for r greater than A, and these two have to match here.

There might be cases in which there is a delta function precisely here, like in the p set, in which case you know that with a delta function, the derivatives have to match in a funny way. There's a discontinuity in the derivatives, and the wave functions do have to match.

So let's assume, in general, that there's no delta function there. And let's do that case. If you have a delta function you could do a similar case. And the way to do this is actually to match the wave functions and the derivatives. So matching at r equal a. So you must have Rlk of a is equal to Al Jl of ka plus Bl nl of ka.

The derivatives must also match. Let me use primes to indicate derivatives with respect to the argument. Most people use that for primes. So the derivative of this quantity Rlk a prime-- so that means the derivative of this function of r evaluated at a. And you differentiate with respect to a in order to get the units not to change. You can multiply by an a. You have to multiply then by an a on the right-hand side when I take the derivative.

Now, when I take the derivative here with respect to r, I take the derivative of J with respect to the argument, and then the relative of the argument with respect to r. That takes out the k out. Since I put an a, I get a ka times Al J prime l at ka. The derivative of J with respect to the argument evaluated at ka plus Bl n prime l at ka.

So here it is. I've matched the function and the derivatives. So I'm giving you an algorithm to do it any time and all times. At this point, the right thing to do is to divide the equations because that gets rid of a lot of things that we are not interested in. We're not interested in anything except the phase shift. That's all we want, the phase shift.

So how are we going to get the phase shift? Well, the phase shift comes from Bl over Al. So let's form the ratio here. So I'll form the ratio aRlk prime of a over Rlk at a. And then I have ka. That's still there.

And then let me divide-- I'm going to divide these two things, but then I can divide the numerator and the denominator by Al. So here I'll have the ratio of Jl prime ka plus Bl over Al n prime l of ka over-- my equation came out a little unbalanced, but it's not so bad. ka. Let's lower this.

OK. Numerator. And here you get Jl of ka plus Bl over Al nl of ka. But this is nothing but ka and the ratio Jl prime at ka minus tan delta k-- no, delta l. That's what the ratio Bl over Al is. n prime l of ka over Jl of ka minus tan delta l eta. Not eta. It's n, actually. n of ka.

That's it. Actually, we've solved the problem. Why? Because this number is supposed to be known because you've solved the equation. You did solve the equation, the Schrodinger equation, from r equals 0 to r equal a. It's completely solved.

So that number is known. The Bessel functions, spherical Bessels, are known. And all you need to find is tangent delta, and that equation determines it. Should I write the formula for tangent delta? OK. Let's write. Let's write it.

Yeah. OK. One minute to write it. It's just solving from here. Tan delta of l is equal J prime l at ka minus Rl prime at a over kRl of k of a Jl of ka over nl prime at ka minus Rl prime of a over kRl of a, that same constant nl of ka. So that's solving for tangent delta from this side equal to this side.

This shows that, in principle, once you have the solution from 0 to a, you have solved the problem of the phase shift. You know the phase shift. It's a matter of just calculating them. Might be a little messy. But if your calculation is important and this is a problem you really want to solve, you will be willing to look at the spherical Bessel functions, which, in fact, are trigonometric functions times powers. They're not as difficult as the ordinary Bessel functions.

So these are easy to work with. And these things are, many times, done numerically. For a given potential, you calculate the first 10 phase shifts, and you get a solution that is very accurate. So there's a lot of power in this formulation. So you have a solution. You have the phase shifts.