MITOCW | L19.2 Energy eigenstates: incident and outgoing waves. Scattering amplitude

OK. So that's our introduction to the subject. Now, we have to get going. We have to explore how to set up this scattering problem. So equations that we need to solve. Well, what are the equations? We will have a Hamiltonian, which is p squared over 2m plus a potential v of r. Need not be yet a central potential. As usual, we will think of a wave function that depends on r and t that will be written as a psi that depends on r times e to the minus iet over h bar. An energy eigenstate. And then the equation that you have to solve, the time independent the Schrodinger equation, becomes minus h squared over 2m Laplacian plus v of r acting on psi of r is equal to e psi of r.

So these are the equations that you have seen already endlessly in quantum mechanics. These are equations we write to get warmed up, and we just repeat for ourselves that we have a Hamiltonian in the picture where the particles are scattering off of a central potential. Not a central potential, in fact, just often of a potential. Work with energy eigenstates, and these are going to be the equation for the energy eigenstates. Now, let's write the first picture of the scattering process as some sort of target here or potential and particles that come in and scatter off of this potential. So we're looking for energy eigenstates. And we will try to identify our energy eigenstates, and what we're going to assume is that this potential is finite range as well.

Range. Finite range we can deal with potentials that fall off relatively fast. Already, the coolant potential doesn't fall off that fast, but potentials that fall faster and the cooler potentials are potentials that are just localized, which is pretty common if you have an atom and you scatter things off of it. If it's a neutral atom, the potential due to the atom is zero outside the atom, but as soon as you go inside the atom you start to experience all the electrical forces that are due to the nucleus and the electrons. So a finite range potential, and we're going to think of solutions that-- OK, away from the finite range our plane waves, solutions of constant energy specified with perhaps some momentum, so we will think of e as h squared k squared over 2m.

This is a way of thinking of the energy of a given energy eigenstate. So it's another label for the energy. This one, it looks like I've done something but I haven't done much except to begin an intuition process in your head in which somehow these are going to be related to energy eigenstates that have some momentum as they propagate all over space. So if I write that, I could just simply put this on the left hand side and get an equation that is kind of nicer. Minus h squared over 2m Laplacian squared plus k squared plus v of r psi of r equals zero. OK, it's not really a matter scattering of solving this equation at this moment. There is infinitely many solutions of this equation, and most of them may not be relevant for us.

We're not trying to find every solution of this equation. We're trying to find solutions that have something to do with physics, and you've done that when you had potentials in one dimension. And that intuition is going to prove invaluable. So when you have a potential in one dimension, you didn't say, OK I'm going to find all the energy eigenstates. You said, let's search for things that are reasonable and physically motivated, so you put in a wave that was moving in and you said, OK, this wave is a solution until it reaches this point where it just stops being a solution and you need more. If you put in this wave, you will generate a reflected wave and a transmitted wave. Those two waves are going to be generated, and then you write an ansatz for this wave, some coefficient a e to the ikx, b e to the minus ikx, c e to the other, and then you solve your equation.

So we need to do the same thing with this kind of equation and this kind of potential. We have to set up some sort of situation where we have the physics intuition of a wave that is coming in and then whatever the system will do to that wave to upgrade it into a full solution. So that's what we want to do here in analogy to that thing. This could be called the incoming wave, and this whole thing the reflected and the transmitted could be called the scattered wave, the thing that gets produced by the scattering process. So if you had that there is lots of solutions of that equation, if v was identical to zero, if you had no potential, you could have lots of solutions, because in fact if the potential is not zero, plane waves are always solutions without any potential.

Particles that move as plane waves so if equal zero, plane waves of the form psi equal e to the ikx are solutions with k equal the square root of k dot k. Or with k squared equals k dot k. Plane waves are always solutions of this equation. So if plane waves are solutions of this equation for v equal to zero, this is the same thing as saying here that ae to the ikx is a solution of this equation as long as you don't hit the potential. So here, we're going to do something quite similar. We're going to say that we're going to put in an incident wave function, and I will instead of writing psi, I will call it 5x. The incident wave function 5x is going to be just e to the ikz.

So it's a wave function. I call it phi to distinguish it from psi. Psi in general is a full solution of the Schrodinger equation. That's our understanding of psi. So phi, it reminds you that well, it's some wave function. I'm not sure it's a solution. In fact, it probably is not the solution as soon as you have the potential different from zero. So is this common, we forget that ksv equals zero, and now we put an incident wave function which is of this form. This solves the equation as long as you're away from the potential. This is true, it's a solution of the Schrodinger equation away from v of r. Away from v of r means wherever v of r is equal zero, you have a solution.

Nevertheless, so if we call the range of the potential-- let's call the range of the potential finite range a-- that is to take that if there is an origin here up to a radius a, there is some potential and beyond the radius a the potential vanishes. So this definitely works. It's all ks away from vr or as long as r is greater than a for whatever value of z you take. This is fine, so here is our wave, and now this is just the incident wave. This is not going to be the whole solution. Just like in the one dimensional case, there must be more. So what is there more?

And our challenge to begin with this problem is to set up what else could there be. So looking at it, you'd say, all right. So the thing comes in. If there is scattering, particles are sometimes going to go off in various directions. So the outgoing wave, here there were outgoing waves reflected and transmitted. The outgoing wave in the three dimensional scattering problem should be some sort of spherical wave moving away from r equals zero, which is the origin. That should be the other wave that I would write. So my ansatz should be that there is some sort of spherical wave that is moving away. So to complete this with a spherical outgoing wave. So while here this is a plane wave moving in the direction of the vector k, if I want to write the spherical wave, I would write e to the i just kr.

E to the ikr is spherically symmetric, and it propagates radiantly out. If you remember as usual that you have e to the minus iet over h bar, so you have kr minus et, that is a wave that propagates radially out. So maybe this is kind of the scattered wave. This e to the ikr moving out everywhere would be your scattered wave. If that is the scattered wave, remember the scattered wave should solve the Schrodinger equation. In fact, the sum of these two should solve the scattering equation. On the other hand, we've seen that this solves it as long as you're away from the potential, and therefore this should also solve it if you're away from the potential. So I ask you, do you think this solves the Schrodinger equation when you're away from the potential?

Would e to the ikr solve the equation? Well for that, you would need that if you're away from the potential, do you have Laplacian of e to the ikr roughly equals zero? Is that true? Would it solve it. No. It doesn't solve it. Doesn't even come close to solving it. It's pretty bad, but the reason it's bad is physically clear. This wave as it expands out must become weaker and weaker so that the probability flux remains constant. You know, you don't want an accumulation of probability between a shell at one kilometer and a shell at two kilometers. So whatever flux is going out from the shell in one kilometer should be going out of the bigger shell. So therefore, it should fall off with r.

Oh, I'm sorry. I didn't write this well. So if you are going to have this to be a solution of the Schrodinger equation outside of vr equals zero, you should have Laplacian k squared of this thing equal to zero. So if this is equal to zero and the potential is equal to zero, the whole thing is equal to zero. So we need this to hold. But even this one, of course, is not true. It is just absolutely not true. The one that works is the following. Laplacian plus k squared of e to the ikr over r is equal to zero for r different from zero. This is a computation I think you guys have done before when you were studying the Hermiticity of p squared. You ended up doing this kind of things.

This Laplacian produces a delta function at r equals zero, but r equals zero is not the place we're interested in. We're trying to find how the waves look away from the scattering center. So we need this to hold away from r equals zero, in fact, bigger for r bigger than a. One way of checking this kind of thing is to remember that the Laplacian of a function of r is in fact one over r d second dr squared r times f. That's a neat formula for the Laplacian of a function that just depends on r. If it depends on theta and phi, it's more complicated. But with this function, it becomes a one line calculation to do this, and the r here is just fantastic, because by the time you multiply by r this function is just exponential.

You take two derivatives, you get minus k squared, and then the r gets canceled as well and everything works beautifully here. And so this equation holds OK. And then we do have a possible scattered wave. So we're almost there. We can write the scattering wave size scattering of x could be e to the ikr over r and then leave it at that. But this would not be general enough. There is no reason why this wave would not depend also on theta and phi. Here is the z direction, and there's points with angle theta, and if you rotate it with some angle phi here, and therefore this function could as well depend on theta and phi.

So we'll include that factor f of theta and phi. Now you would say, look, you have a nice solution of the Schrodinger equation already here, and what should you be doing? Why do you add this factor? With this factor, it may not be anymore a solution of the Schrodinger equation. The Schrodinger equation is going to have a Laplacian and it's going to be more complicated. Well, this is true and we will see that better soon, but this will remain an approximate solution for r much bigger than a is the leading term of the solution, leaving term of the solution. So if you have a leading term of a solution here, this is all you want. You are working at r much bigger than a, and your whole wave and finally be written.

So your full wave psi, or your full energy eigenstates psi of rt, is going to be equal, approximately equal to e to the ik-- well, I'll write it this way. Phi of r, that incident wave we wrote-- I wrote if x there, I'm sorry. Plus psi scattering of-- I should decide x or r. Let's call this r. And I should call this r. Psi of r plus psi scattering of r, and it's therefore equal to e to the ikz plus f of theta phi e to the ikr over r, and this is only valid for r much bigger than a. Far away. This had an analog in our problem in one dimension. In one dimension, you set up a wave and you put here the wave and there is a reflected wave, and there's a transmitted wave, and in setting the problem, say this is valid far to the left, far to the left meaning at least to the left of the barrier, this is valid to the right. And these were exact solutions in this region here.

You can't do that well, but you can do reasonably well, you can write solutions that are leading term accurate as long as you are far away. So this is the first step in this whole process in which we are setting up the wave functions that is the most important equation. This is the way we're going to try to find solutions. Now, when you try to find solutions at the end of the day, you will have to work in the region r going to zero. So sooner or later we'll get there. But for the time being, we have all the information about what's going on far away, and from there we can get most of what we need. In particular, this f of theta and phi is the quantity we need to figure out. This f of theta and phi is called the scattering amplitude. F of theta and phi called scattering amplitude.