Quantum Physics III (8.06) Spring 2005
Assignment 1

Feb 2, 2005
Due Feb 9, 2005

Readings

The reading assignment for the first three lectures of 8.06 is:

- Griffiths, Ch. 5.3

- Cohen-Tannoudji, Ch. XI Complement F.

Useful Facts

- 1 atmosphere = 1.01×10^6 dynes/cm^2 (cgs)

- $M_{\text{sun}} = 1.99 \times 10^{33}$ grams

- 1 MeV = 1.602×10^{-6} ergs (cgs)

Problem Set 1

1. **cgs Units for Mechanical and Electromagnetic Quantities (12 points)**

   There are several slightly different ways for generalizing cgs units to electro-magnetism. The differences revolve around factors of c and 4π. We use units where Gauss' Law reads $\vec{\nabla} \cdot E = 4\pi\rho$ and where a factor of c is introduced into the Lorentz force law so that $\vec{E}$ and $\vec{B}$ have the same units. This system is known as "Gaussian units". We can just think of it as cgs extended to include electromagnetism.

   Find the cgs units — which must take the form gm^{a}cm^{b}sec^c — for the following quantities:

   (a) Force
   (b) Energy
   (c) Surface tension
   (d) Pressure
   (e) Energy flux (energy per unit area per unit time)

(f) Electric charge

(g) Electric current

(h) Electric field

(i) Magnetic field

(j) Magnetic flux

(k) Voltage

(l) Conductance

(m) Conductivity

(n) Resistance

(o) Resistivity

(p) Inductance

(q) Magnetic moment

You should be able to deduce the units of electromagnetic quantities from the physical laws that define these quantities in the cgs system:[1]

$$\vec{F}_{12} = \frac{e_1 e_2}{r_{12}^2} \hat{r}_{12} \qquad \text{Coulomb's Law}$$

$$\vec{F} = e\vec{E} + \frac{e}{c}\vec{v} \times \vec{B}$$

$$V = \int \vec{dl} \cdot \vec{E}$$

$$V = IR \qquad \text{Ohm's law}$$

$$R = \rho \times \text{length/area}$$

$$V = -\frac{L}{c}\frac{dI}{dt} \qquad \text{Faraday's law}$$

$$\vec{\nabla} \times \vec{B} = \frac{4\pi}{c}\vec{j} \qquad \text{Ampere's Law}$$

And, recall that the interaction energy of a magnetic moment $\vec{\mu}$ in a magnetic field is $\vec{\mu} \cdot \vec{B}$.

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[1]If you are curious about the factor of $1/c$ that appears in Faraday's law, look at Jackson page 820-821 for further discussion.

2. **Electromagnetic energy density, momentum density and energy flux (3 points)**

   If $\vec{E}$ and $\vec{B}$ are electric and magnetic fields, show that:

   (a) $\vec{E}^2$ and $\vec{B}^2$ have the units of energy density,

   (b) $\frac{1}{c}\vec{E} \times \vec{B}$ has units of momentum density,

   (c) $c\vec{E} \times \vec{B}$ has units of energy flux.

3. **Natural Units (3 points)**

   First, you should verify Equation (12) in the notes on Natural Units, reproduced here:

$$
\begin{aligned}
[\text{mass}] &= \text{eV}c^{-2} \\
[\text{time}] &= (\text{eV})^{-1}\hbar \\
[\text{length}] &= (\text{eV})^{-1}\hbar c \\
[\text{momentum}] &= \text{eV}c^{-1} \\
[\text{force}] &= (\text{eV})^2\hbar^{-1}c^{-1} \\
[\text{pressure}] &= (\text{eV})^4\hbar^{-3}c^{-3} \\
[\text{charge}^2] &= \hbar c \\
[\text{magnetic field}] &= (\text{eV})^2\hbar^{-3/2}c^{-3/2} \quad ,
\end{aligned}
\tag{1}
$$

   Do not turn this in.

   For credit, work out the natural units for power, magnetic flux, and magnetic moment.

4. **Magnetic Moments (6 points)**

   A particle's magnetic moment is linearly proportional to its electric charge, e, and to its spin, $\vec{S}$. Write an expression for a particle's magnetic moment in terms of these quantities and its mass m and the fundamental constants $\hbar$ and c. Call the dimensionless constant of proportionality g.

   For an electron, $g \simeq 2$ and its spin is quantized to $\pm\hbar/2$. What is the difference between the energies of electrons with their spins parallel to and antiparallel to a magnetic field of 10^5 gauss? Express your answer in electron volts.

5. **The Bag Pressure (16 points)**

The mass, force and energy scales that characterize the physics of quarks are gigantic when expressed in terms of every days units. Here is a problem to show the size of the forces that are at work inside protons and neutrons.

The proton and neutron ("nucleons" for short) are made of three almost massless quarks. The rest mass of the nucleon is approximately 940 MeV. (Note: $c = 1$.) The quarks are confined to the interior of nucleons because it takes work to "open up" a region of space in which they can be present. (Another way of saying this: the presence of the quarks disturbs the vacuum, and thus costs energy.) The work that must be done in order to make a space in which quarks can live is parametrized by a constant known as the "bag constant", B, which has units of energy per unit volume. That is, the work needed to make a "quark bag" of volume V is BV.

A simple model of the nucleon (invented here at MIT by Prof. Jaffe and his collaborators in the 1970's) treats it as a spherical bag of radius R, with a rest energy (i.e. rest mass) that includes only two terms: a) the work done to open up a bag, and b) the zero point kinetic energy of the massless quarks confined within the bag. The three massless quarks have kinetic energy $3p$, where p is the mean momentum of a quark within the bag. (Again, note that $c = 1$.) The uncertainty principle tells us that $p \approx 1/R$, (Note: $\hbar = 1$.) So, the rest energy of the nucleon can be written as the sum of the kinetic zero point energy and the bag energy:

$$E(R) = \frac{3}{R} + B\frac{4\pi R^3}{3} \; . \tag{2}$$

In this model, the radius of the nucleon R_0 is the value of R that minimizes $E(R)$.

In the following, you will want to use the conversion factors given in (9) and (13) in the notes on Natural Units.

(a) Restore the factors of $\hbar$ and c to Eq. (2) to make it dimensionally correct. Then set $\hbar = c = 1$ once again for the rest of the problem.

(b) Find R_0. It will be a function of B. Then, find the rest mass of the nucleon as a function of B by substituting R_0 back into Eq. (2).

(c) Given the mass of the nucleon, $M = 940$ MeV, find the numerical value of B. Evaluate B in MeV/fm^3. [Here, 1 MeV= 10^6 eV and 1 fm= 10^{-13} cm.]

(d) Show that energy density and pressure have the same units. (You can use results from Problem 1.)

(e) Evaluate B in atmospheres. This is the pressure that quarks must fight against to open up a bag.

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(f) Compute the radius R_0 in fm.

[In reality, the proton does not have a sharp edge, and so R_0 cannot be defined literally as we did in this problem, and the bag model cannot be used to describe all of the properties of the proton. Still, a reasonable characterization of much experimental data is that the radius of the proton is a little less than 1 fm. The fact that your calculation predicts a value of R_0 that is *so* close to this is a coincidence, given that we used the crude estimate $p \approx 1/R$ and given that the model itself is oversimplified in many ways, including treating nucleons as if they have a sharp edge. Nevertheless, the fact that your calculation which used M as an input was able to predict a value of R_0 that is in the right ballpark supports the basic picture of the physics described by the bag model. In particular, it supports the notion that the quarks inside a nucleon are themselves much lighter than 1/3 the mass of the nucleon.]

6. **Quantum Gravity (10 points)**

When we consider gravitational forces in the cgs system, we introduce a proportionality factor, Newton's constant G_N, defined via

$$\vec{F}_{12}(r) = G_N \frac{m_1 m_2}{r^2} \hat{r} \tag{3}$$

Then G_N measures the gravitational force in dynes between two objects each of mass 1 gm, separated by a distance of 1 cm. Experimental measurement tells us that $G_N = 6.67259(85) \times 10^{-8}$ cm^3/gm sec^2.

(a) Compute the value of G_N in natural units (see the lecture notes).

(b) Calculate the gravitational interaction energy between two electrons separated by their compton wavelength. Express the result as a fraction of their rest mass. Give the result in terms of fundamental constants and also numerically. This is a measure of the strength of gravity which can be compared with the strength of electromagnetism (measured by α).

(c) If you get very close to a very massive pointlike object, relativistic effects modify the predictions of Newtonian gravity and it is necessary to use general relativity. Estimate the length scale of relativistic gravity for a point mass m by combining m with G_N and c to obtain a length. This length is one half the "Schwarzschild radius", r_s, of the mass m. In general relativity if a mass m is concentrated inside its Schwarzschild radius, it forms a black hole. Evaluate the Schwarzschild radius for an object of mass 1 gm. Evaluate it for the sun.

(d) Quantum effects become important when we look at distances of order the compton wavelength of a particle. If a particle's Compton wavelength was

the same order as its Schwarzschild radius, then it would be necessary to understand quantum gravity to describe physics at those distances. Find the mass of such a particle. (This mass is called M_{Planck}.) What is its Compton wavelength? (This length scale is called L_{Planck}.)

(e) One of the biggest recent discoveries in science is that the expansion of the universe is accelerating. This can be described, but not really understood, by reintroducing what Einstein called the cosmological constant Λ into the equations of general relativity. Λ is in many ways like a negative bag constant (see previous problem) for the whole universe. (Speaking *very* loosely: With a positive bag constant it costs energy to make a proton. With a negative bag constant, on the other hand, the equations favor creating more and more bag. In the case of cosmology, this means more and more universe, and the expansion of the universe accelerates.) Once you realize that there could be a Λ in Einstein's equations, you have to ask how large should it be? The simplest answer to this is that within any relativistic quantum theory of gravity, like for example string theory, it would seem that Λ should be of order $M_{\text{Planck}}/L^3_{\text{Planck}}$. The measured value of Λ — in other words, the Λ that must be introduced in order to parametrize the recently observed accelerating expansion of the universe — is $\Lambda \approx 2000$ eV/cm³. What is the ratio of the observed Λ to $M_{\text{Planck}}/L^3_{\text{Planck}}$? Another way of saying this: What is the observed Λ in Planck units? Understanding the smallness of Λ is one of the biggest challenges in theoretical physics.

7. **Eigenstates of the "Translation" Operator (10 points)**

Consider an infinite dimensional Hilbert space with orthonormal basis states that we will call $|n\rangle$ where n is an integer running from $-\infty$ to $+\infty$.

The Hamiltonian for the system is

$$H = \sum_{n=-\infty}^{n=+\infty} \left[E_0 |n\rangle\langle n| \ + \ \Delta |n\rangle\langle n+1| \ + \ \Delta |n+1\rangle\langle n| \right].$$

Note that the basis states are not energy eigenstates.

Define the "translation" operator T by:

$$T|n\rangle = |n+1\rangle$$

(a) Show that H and T commute.

(b) Find the state which is an eigenstate of T with eigenvalue $\exp(-i\theta)$. (Call this state $|\theta\rangle$.)

(c) Show that $|\theta\rangle$ is an eigenstate of H. What is its energy?