Quantum Physics II (8.05) Fall 2013
Assignment 7

Massachusetts Institute of Technology
Physics Department
October 24, 2013

Due 1 November 2013
3:00 pm

Problem Set 7

1. **Spin in a time-varying magnetic field** [10 points]

   A spin is placed on an uniform but oscillating magnetic field
   $$\vec{B} = B_0 \cos(\omega t) \vec{e}_z \, .$$
   The spin is initially in an eigenstate of S_x with eigenvalue $\hbar/2$.

   (a) Find the unitary operator $\mathcal{U}(t)$ that generates time evolution. Note that the Hamiltonian is time-dependent but $[H(t), H(t')] = 0$.

   (b) Calculate the time evolution of the state and describe it by giving the time-dependent angles $\theta(t)$ and $\phi(t)$ that define the direction of the spin.

   (c) Find the time dependent probability to find the spin with $S_x = -\hbar/2$.

   (d) Find the largest value of ω that allows the full flip in S_x.

2. **Heisenberg operators for spin** [5 points]

   Consider the time-independent Schrödinger Hamiltonian for a spin in a uniform and constant magnetic field of magnitude B along the z-direction:
   $$H = -\lambda B \, S_z \, .$$
   Here λ is the (real) constant that relates the dipole moment to the spin. Find the explicit time evolution for the Heisenberg operators $\hat{S}_x(t)$, $\hat{S}_y(t)$, and $\hat{S}_z(t)$ associated with the Schrödinger operators S_x, S_y, and S_z.

3. **The Heisenberg Picture and Newton's Laws** [10 points]

   (a) Consider the Hamiltonian $\hat{H} = \hat{p}^2/(2m) + V(\hat{x})$ and derive the Heisenberg equations of motion for $\hat{x}_H(t)$ and $\hat{p}_H(t)$. Use your results to obtain Ehrenfest's theorem
   $$\frac{d}{dt}\langle \hat{x} \rangle = \frac{\langle \hat{p} \rangle}{m} \, , \qquad \frac{d}{dt}\langle \hat{p} \rangle = -\langle V'(\hat{x}) \rangle \, , \tag{1}$$
   where $\langle \hat{x} \rangle = \langle \psi, 0 | \hat{x}_H(t) | \psi, 0 \rangle = \langle \psi, t | \hat{x} | \psi, t \rangle$ etc. Combine them to derive an equation for $\frac{d^2}{dt^2}\langle \hat{x} \rangle$. Explain the conditions on the potential such that this equation reduces to the classical Newton's Law.

(b) Consider a free particle in a normalized state whose average position and momentum at $t = 0$ are x_0 and p_0. Use Ehrenfest's theorem to determine $\langle \hat{x} \rangle$ as a function of time.

(c) Now imagine that this particle has a charge q, and consider applying an electric field that varies with time, so $V(\hat{x}) = qE_0\,\hat{x}\sin(\omega t)$. Demonstrate that now $[\hat{H}(t_1), \hat{H}(t_2)] \neq 0$ for $t_1 \neq t_2$. Look back at the steps underlying the derivation of Eq. (1) and explain why it still holds.

(d) Find $\langle \hat{x} \rangle$ as a function of time for the situation in part (c).

4. **Virial theorem** [10 points]

Consider a Hamiltonian for a particle in three dimensions under the influence of a central potential:

$$H = \frac{\vec{p}^{\,2}}{2m} + V(r)\,,$$

as well as the Schrödinger operator $\Omega \equiv \vec{r} \cdot \vec{p}$. We let $\Omega_H(t)$ denote the associated Heisenberg operator.

(a) Use the Heisenberg equation of motion to calculate the time rate of change $\frac{d}{dt}\Omega_H(t)$. Your answer for the right-hand side should be in terms of the Heisenberg operators $\vec{p}_H^{\,2}$, $\vec{r}_H$, derivatives of $V(r_H)$, and constants.

(b) Consider a stationary state $|\Psi, t\rangle$ and *any* Heisenberg operator $\mathcal{O}_H(t)$ arising from a time-independent Schrödinger operator. Explain carefully why

$$\langle \Psi, 0| \frac{d}{dt}\mathcal{O}_H(t) |\Psi, 0\rangle = 0\,.$$

(c) Use your results from (a) and (b) to show that for a potential $V(r) = c/r^k$, with c constant and k a positive integer

$$\langle T \rangle = -\frac{k}{2}\langle V \rangle\,.$$

Here the expectation value is taken on a stationary state, T denotes the kinetic energy operator $\frac{\vec{p}^{\,2}}{2m}$, and V denotes the potential.

5. **Time Evolution in the Heisenberg Picture** [10 points]

In this problem we'll study the time evolution of a wave packet acted upon by a constant force. This is a case where the Schrödinger equation is hard to solve, but the Heisenberg equations of motion for the time dependence of operators can be solved easily and quite a bit can be learned about the motion.

Suppose a quantum particle is described the Hamiltonian,

$$\hat{H} = \frac{\hat{p}^2}{2m} + g\hat{x}\,,$$

which corresponds to the particle subject to a constant force $F = -\frac{dV}{dx} = -g$.

(a) Use the Heisenberg equations of motion to show that the Heisenberg operators $\hat{x}_H(t)$ and $\hat{p}_H(t)$ obey an analog of Newton's law $F = ma$. Integrate the Heisenberg equations of motion to obtain $\hat{x}_H(t)$ in terms of $\hat{x}_H(0) = \hat{x}$ and $\hat{p}_H(0) = \hat{p}$.

(b) Suppose that at $t = 0$ a particle has coordinate space wavefunction,

$$\langle x|\psi\rangle = \psi(x) = Ne^{-\frac{x^2}{2\Delta^2}},$$

where N is a constant that normalizes ψ to unity. Compute $\langle\psi|\hat{x}_H(t)|\psi\rangle$ and show that it behaves classically.

(c) Compute the squared uncertainty in x, namely $(\Delta x(t))^2 = \langle\hat{x}_H^2(t)\rangle - \langle\hat{x}_H(t)\rangle^2$. Show that $(\Delta x(t))^2$ grows quadratically with time,

$$(\Delta x(t))^2 = (\Delta x(0))^2 + \lambda t^2$$

and find the coefficient λ. How does the spreading of the wavepacket depend on the value of g?

6. **Shifted harmonic oscillator** [10 points]

A quantum harmonic oscillator perturbed by a constant force of magnitude F in the positive x direction is described by the Hamiltonian

$$H = \frac{\hat{p}^2}{2m} + \frac{1}{2}m\omega^2\hat{x}^2 - F\hat{x}\,.$$

Note that if $\hat{x}$ and $\hat{p}$ satisfy $[\hat{x}, \hat{p}] = i\hbar$, we also have $[\hat{x} - x_0\,,\,\hat{p}] = i\hbar$, for any constant x_0, demonstrating that $\hat{y} \equiv \hat{x} - x_0$ and $\hat{p}$ form a pair of conjugate variables.

(a) Find the ground state energy of H. What is $\langle\hat{x}\rangle$ in the ground state?

(b) The ground state $|0'\rangle$ of the H can be written as

$$|0'\rangle = Ne^{\alpha\hat{a}^\dagger}|0\rangle,$$

where $\hat{a}^\dagger$ and $|0\rangle$ are respectively the raising operator and ground state of the *unperturbed* $F = 0$ Hamiltonian. Find the real number α. Hint: consider operators $\hat{a}_y$ and $\hat{a}_y^\dagger$ based on $\hat{y}$ and $\hat{p}$.

7. **Wavefunction for a coherent state** [10 points]

Consider the unit-normalized coherent state

$$|\alpha\rangle \;=\; e^{\alpha\hat{a}^\dagger - \alpha^*\hat{a}}|0\rangle$$

where α is a complex number parameterized as

$$\alpha = \frac{x_0}{\sqrt{2}d} + i\frac{p_0 d}{\sqrt{2}\hbar}\,, \quad \text{with} \quad d = \sqrt{\frac{\hbar}{m\omega}}\,, \quad x_0, p_0 \in \mathbb{R}\,.$$

Calculate the wavefunction $\psi_\alpha(x) = \langle x|\alpha\rangle$. Your answer for this wavefunction should come out manifestly unit-normalized and can be written in terms of the function that represents the ground state of the oscillator.

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