Quantum Physics II (8.05) Fall 2013
Assignment 1

Massachusetts Institute of Technology
Physics Department
September 4, 2013

Due Friday September 13, 2013
3:00pm

Announcements

- Please put you name and section number at the top of your problem set, and place it in the 8.05 box labeled with your section number near 8-395 by 3pm Friday.
- Recommended Reading for the first week: Shankar, sections 5.2, 5.3, and 5.6. Griffiths, sections 2.1, 2.2, 2.5, and 2.6.

Problem Set 1

1. **Properties of a wavefunction.** [10 points]

   A particle of mass m in a one-dimensional potential $V(x)$ has the wave function

   $$\psi(x) = Nx \exp\left(-\frac{1}{2}\alpha x^2\right), \quad \alpha > 0\,.$$

   (a) Normalize $\psi(x)$ to determine N. What is $\langle \hat{x} \rangle$? What is $\langle \hat{x}^2 \rangle$?

   (b) What is $\langle \hat{p} \rangle$? What is $\langle \hat{p}^2 \rangle$?

   (c) Is $\psi(x)$ a position eigenstate? Is $\psi(x)$ a momentum eigenstate? Explain.

   (d) Suppose that $V(x) = 0$. What is $\langle \hat{H} \rangle$?

   (e) Suppose that nothing is known about $V(x)$, but $\psi(x)$ is an energy eigenstate. Find the potential $V(x)$ and the energy eigenvalue E, assuming $V(0) = 0$. Could $\psi(x)$ be the ground state wavefunction for the particle?

2. **Energy must exceed the minimum value of the potential.**[1] [5 points]

   Consider the time-independent Schrödinger equation for a particle of energy E in a potential $V(x)$, with $x \in (-\infty, \infty)$:

   $$\frac{d^2\psi}{dx^2} = \frac{2m}{\hbar^2}\left[V(x) - E\right]\psi(x)\,. \tag{1}$$

[1] A variation on Griffiths 2.2.

Without loss of generality one can assume that $\psi(x)$ is real. Assume the potential is bounded below,

$$V(x) \geq V_{\text{min}}\,, \quad \text{for all} \quad x,$$

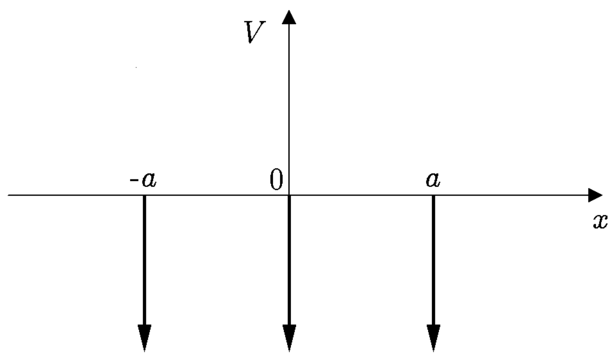
where V_{min} is the minimum value of the potential.

Prove that $E > V_{\text{min}}$ for *normalizable* solutions to exist. To do this, assume $E \leq V_{\text{min}}$ and try using equation (1) and integration to reach a clear contradiction.

3. **Three Delta Functions** [15 points]

A particle of mass m moves in one dimension, subject to a potential energy function $V(x)$ which is the sum of three evenly spaced attractive delta functions:

$$V(x) = -V_0\, a \sum_{n=-1}^{1} \delta(x - na)\ \text{, where } V_0 > 0,\ a > 0 \text{ are constants.}$$



(a) Calculate the discontinuity in the first derivative of the wavefunction at $x = -a$, 0, and a.

(b) Consider the possible number and locations of nodes in bound state wavefunctions for this system.

(i) How many nodes are possible in the region $x > a$?

(ii) How many nodes are possible in the region $0 < x < a$?

(iii) Can there be a node at $x = a$?

(iv) Can there be a node at $x = 0$?

(c) For arbitrarily large V_0, how many bound states are there? Sketch them qualitatively.

(d) Derive the equation that determines the energy for the lowest energy antisymmetric bound state. Find the minimum value of V_0 for the bound state to exist.

4. **Estimates on the finite square well** [10 points]

   Consider the finite square well potential in section 2.6 of Griffiths:

$$V(x) = -V_0 \quad \text{for} -a \leq x \leq a, \quad \text{and} \quad V(x) = 0 \quad \text{for } |x| > a\,.$$

   (a) *Number of bound states for deep well.* Assume that the well is sufficiently deep and/or wide so that z_0, defined as

$$z_0 \equiv \frac{a}{\hbar}\sqrt{2mV_0}\,,$$

   is a large number. Find an estimate for the number of bound states in this well using the result that the k-th bound state has $k-1$ nodes. Confirm that your result is a good approximation by comparing with Figure 2.18 in the book.

   (b) *Energy of the bound state for a shallow well.* Assume now that the potential is very shallow and/or narrow so that z_0 is a very small number and as a result there is just one bound state. Use the relevant equations of the problem (see Griffiths) to estimate the energy E of this state in terms of V_0 and z_0 (*i.e.* find the leading term of the energy in the expansion in terms of z_0, as $z_0 \to 0$).

5. **Expectation value $\langle \hat{p} \rangle$ of the momentum.** [5 points]

   (a) A particle's coordinate space wavefunction is square-integrable and real up to an arbitrary multiplicative phase:

$$\psi(x) = e^{i\alpha}\phi(x)\,,$$

   with α real and constant and $\phi(x)$ real. Prove that the expectation value of the momentum is zero.

   (b) Consider instead the wavefunction

$$\psi(x) = \phi_1(x) + e^{i\alpha}\phi_2(x)\,,$$

   where $\phi_1(x)$ and $\phi_2(x)$ are each real and square-integrable. What is $\langle \hat{p} \rangle$? The answer can be expressed as a function of α times an integral that involves the functions ϕ_2 and $d\phi_1/dx$ (or ϕ_1 and $d\phi_2/dx$). For what values of α can we be sure that $\langle \hat{p} \rangle$ is zero without having further information about ϕ_1 and ϕ_2?

   (c) Consider this time the wavefunction

$$\psi(x) = e^{ikx}\phi(x)\,,$$

   with k real and constant and $\phi(x)$ real. Calculate $\langle \hat{p} \rangle$.

6. **Conserved probability current.** [10 points]

   Suppose $\Psi(x,t)$ obeys the one-dimensional Schrödinger equation,

$$-\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2}\Psi(x,t) + V(x)\Psi(x,t) = i\hbar\frac{\partial}{\partial t}\Psi(x,t). \tag{2}$$

   (a) Derive the conservation law for probability,

$$\frac{\partial \rho}{\partial t} + \frac{\partial J}{\partial x} = 0, \tag{3}$$

   where $\rho(x,t)$ is the probability density and $J(x,t)$ is the probability current density

$$\rho(x,t) = \Psi^*\Psi\,, \quad J(x,t) = \frac{\hbar}{m}\,\mathrm{Im}\left(\Psi^*\frac{\partial\Psi}{\partial x}\right). \tag{4}$$

   What are the units of ρ and J?

   (b) Explain why (3) is a conservation law for probability. In order to do so, define

$$P_{ab}(t) \equiv \int_a^b dx\ \rho(x,t)\,,$$

   evaluate $\frac{dP_{ab}}{dt}$ in terms of currents, and interpret your answer. Show then that a wavefunction $\Psi(x,t)$ that is normalized at time t remains normalized at later times.

   (c) In the following we consider stationary states with spatial wavefunctions $\psi(x)$. Compute the probability current J for $\psi(x) = e^{i\alpha(x)}\phi(x)$ where $\alpha(x)$ and $\phi(x)$ are real. Show that

$$\frac{J(x)}{\rho(x)} = \frac{\hbar}{m}\alpha'(x)\,.$$

   Explain why the ratio J/ρ can be viewed as the local velocity of the quantum particle described by $\psi(x)$.

   (d) Consider $\psi(x) = Ae^{ipx/\hbar} + Be^{-ipx/\hbar}$, with A and B complex constants. Calculate $J(x)$. Are there cross terms in J between the left and right-moving parts of ψ?

7. **Griffiths Problem 2.38, p.85** [10 points]