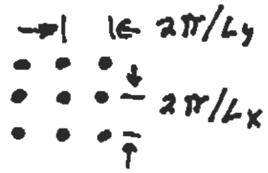
8.044 Solutions Practice Exam #4

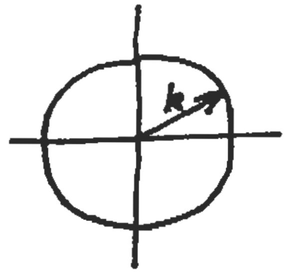
1 a)



$$k\text{-volume/point} = \frac{(2\pi)^2}{L_x L_y}$$

$$\text{points}/k\text{-volume} \equiv D(k) = \frac{L_x L_y}{(2\pi)^2} = \underline{\underline{\frac{A}{(2\pi)^2}}}$$

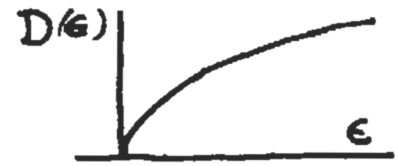
b)



$$\#(\epsilon) = \pi k^2(\epsilon) D(k) \qquad \epsilon = b k^{3/2}$$

$$= \pi \left(\frac{\epsilon}{b}\right)^{4/3} D(k)$$

$$D(\epsilon) = \frac{d\#}{d\epsilon} = \frac{4}{3} \frac{A}{(2\pi)^2} \pi \frac{\epsilon^{1/3}}{b^{4/3}} = \underline{\underline{\frac{A}{3\pi b^{4/3}} \epsilon^{1/3} \quad \text{FOR } \epsilon \geq 0}}$$



c) $$E = \int_0^\infty \epsilon \left(\bar{n} + \tfrac{1}{2}\right) D(\epsilon)\, d\epsilon = \int_0^\infty \epsilon \left(\frac{1}{e^{\epsilon/kT} - 1} + \frac{1}{2}\right) D(\epsilon)\, d\epsilon$$

$$C_A = \left.\frac{\partial E}{\partial T}\right|_A = \int_0^\infty \epsilon \; \frac{\frac{\epsilon}{kT^2} e^{\epsilon/kT}}{(e^{\epsilon/kT} - 1)^2} \; \frac{A}{3\pi b^{4/3}} \epsilon^{1/3}\, d\epsilon$$

$$= \underline{\underline{\frac{Ak}{3\pi b^{4/3}} (kT)^{4/3} \int_0^\infty \frac{x^{7/3} e^x}{(e^x - 1)^2}\, dx}} \propto \underline{\underline{T^{4/3}}}$$

One could also proceed directly from E

$$E = \frac{A}{3\pi b^{4/3}} \int_0^\infty \frac{\epsilon^{4/3}}{e^{\epsilon/kT} - 1}\, d\epsilon + C = \frac{A (kT)^{7/3}}{3\pi b^{4/3}} \int_0^\infty \frac{x^{4/3}}{e^x - 1}\, dx + C$$

$$C_A = \left.\frac{\partial E}{\partial T}\right|_A = \underline{\underline{\frac{7Ak}{9\pi b^{4/3}} (kT)^{4/3} \int_0^\infty \frac{x^{4/3}}{e^x - 1}\, dx}} \propto \underline{\underline{T^{4/3}}}$$

d) There is no energy gap behavior because there is no energy gap. For any kT there are always oscillators with $\hbar\omega < kT$.

2 a) THERE ARE 10 SINGLE PARTICLE STATES INCLUDING SPIN.

\# 3-PARTICLE STATES = # WAYS OF CHOOSING 3 FROM 10 WHEN ORDER DOES NOT MATTER $= 10 \times 9 \times 8 / 3 \times 2 \times 1 = \underline{\underline{120}}$

b) $E = \Delta$: 2 ELECTRONS IN a, 1 IN b OR c $\rightarrow$ 4 STATES

$E = 1.5\Delta$: 2 ELECTRONS IN a, 1 IN d OR e $\rightarrow$ 4 STATES

$$\underline{\underline{Z = 4e^{-\Delta/kT} + 4e^{-1.5\Delta/kT}}} + \cdots$$

c) $E = 4.5\Delta$: 3 ELECTRONS IN d AND e $\rightarrow$ 4 STATES

$E = 4\Delta$: 2 ELECTRONS IN d AND/OR e $\rightarrow$ 6 WAYS

AND 1 ELECTRON IN b OR c $\rightarrow$ 4 WAYS

$\Rightarrow 6 \times 4 = 24$ STATES

$$Z = \cdots + \underline{\underline{24e^{-4\Delta/kT} + 4e^{-4.5\Delta/kT}}}$$

d) $\underline{\underline{k \ln 4}}$ e) $\underline{\underline{k \ln 120}}$

f) $\underline{\underline{C(T) \xrightarrow[T \to \infty]{} 0}}$ SINCE THE TOTAL ENERGY HAS AN UPPER BOUND

g-b) $E = 0$: ALL 3 BOSONS IN a $\rightarrow$ 1 STATE

$E = \Delta$: 2 BOSONS IN a, 1 IN b OR c $\rightarrow$ 2 STATES

$$\underline{\underline{Z = 1 + 2e^{-\Delta/kT} + \cdots}}$$

g-c) $E = 4.5\Delta$: 3 BOSONS IN d AND/OR e → 4 STATES
$E = 4.0\Delta$: 2 BOSONS IN d AND/OR e → 3 WAYS
AND 1 BOSON IN b OR c → 2 WAYS
⇒ $3 \times 2 = 6$ STATES

$$Z = \cdots + \underline{\underline{6e^{-4\Delta/kT} + 4e^{-4.5\Delta/kT}}}$$

g-d) $S(0) \to k \ln 1 = 0$

g-f) $\underline{\underline{C(T) \xrightarrow[T\to\infty]{} 0}}$ SINCE THE TOTAL ENERGY HAS AN UPPER BOUND

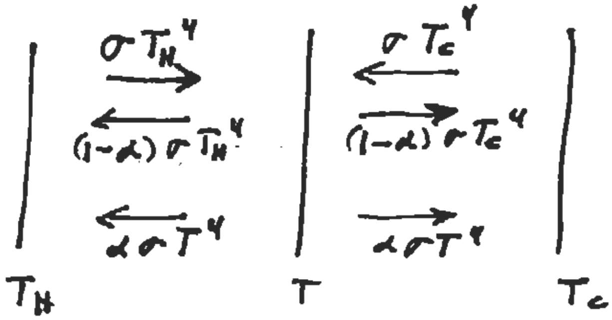
g-a) FOR INFORMATION ONLY!

THE TOTAL NUMBER OF STATES = THE # OF WAYS OF PUTTING 3 SPINLESS BOSONS IN 5 SPATIAL STATES = # WAYS OF PUTTING 3 BALLS IN 5 DIFFERENT BOXES WHEN ORDER DOES NOT MATTER = # WAYS OF ORDERING 3 BALLS AND 4 PARTITIONS WHEN THE ORDER OF THE BALLS DOES NOT COUNT

$$= \frac{7!}{(7-3)!} \times \frac{1}{3!} = \frac{7\cdot 6\cdot 5}{3\cdot 2\cdot 1} = \underline{\underline{35}}$$

IN PARTICULAR THERE ARE $\underline{5}$ WAYS OF PUTTING ALL 3 BOSONS IN THE SAME STATE, $5 \times 4 = \underline{20}$ WAYS OF PUTTING 2 IN 1 STATE AND 1 IN ANOTHER, AND $5 \times 4/2! = \underline{10}$ WAYS OF PUTTING EACH IN A SEPARATE STATE.

3 a)



$$\frac{e}{\alpha} = \sigma T^4$$
$$\Rightarrow e = \alpha\sigma T^4$$

$$T_H^4 - (1-\alpha)T_H^4 - \alpha T^4 = -T_c^4 + (1-\alpha)T_c^4 + \alpha T^4$$
$$\alpha T_H^4 + \alpha T_c^4 = 2\alpha T^4$$
$$\underline{\underline{T^4 = \frac{T_H^4 + T_c^4}{2}}}$$

b) $\sigma\alpha T_H^4 - \sigma\alpha T^4 = \alpha\sigma\left(T_H^4 - T^4\right) = \frac{\alpha\sigma}{2}\left(T_H^4 - T_c^4\right)$

J IN ABSENCE OF SHEET $= \sigma\left(T_H^4 - T_c^4\right) \equiv J_0$

$$J_{SHEET} = \frac{\alpha}{2} J_0 = \underbrace{\left(\frac{1-r}{2}\right)}_{\mathcal{F}} J_0$$

4 THE ENTROPY OF THE IDEAL PARAMAGNET DEPENDS ON H AND T ONLY THROUGH THE RATIO $\eta = \frac{\text{LEVEL SPACING}}{kT} \propto H/T$

ADIABATIC $\Rightarrow$ CONSTANT S $\Rightarrow$ CONSTANT H/T
$\Rightarrow T \propto H$

$$\frac{H_f}{H_i} = \frac{20}{8{,}000} = \frac{1}{400} \Rightarrow T_f = \frac{1}{400} T_i = \underline{\underline{\frac{1}{400} K = 2.5 mK}}$$

NOTE: SINCE THE SPIN SYSTEM IS USUALLY SATURATED AT THE BEGINNING OF SUCH AN EXPERIMENT, ARGUMENTS BASED ON THE CURIE LAW ARE NOT SUFFICIENT.

8.044 Statistical Physics I
Spring 2013