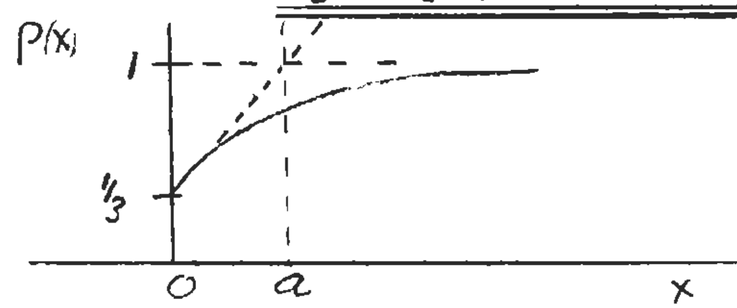
8.044 SOLUTIONS PRACTICE EXAM # 1

1. a/ $P(x) = \int_{-\infty}^{x} p(x')\,dx' = \underline{\underline{0 \text{ for } x<0}}$

$$= \underbrace{\frac{1}{3}}_{\text{FROM } \delta \text{ FUNCTION}} + \frac{2}{3} \underbrace{\int_0^x e^{-x/a}\, d(x/a)}_{\left[-e^{-\xi}\right]_0^{x/a}}$$

$$= \underline{\underline{\frac{1}{3} + \frac{2}{3}\left(1 - e^{-x/a}\right) \qquad x \geq 0}}$$



b/ $\text{Prob.}\ (x>a) = \int_a^\infty p(x)\,dx = \frac{2}{3} \underbrace{\int_1^\infty e^{-\xi}\, d\xi}_{e^{-1}} = \underline{\underline{\frac{2}{3} e^{-1}}}$

c/ $\langle x \rangle = \int_{-\infty}^{\infty} x\, p(x)\,dx = \frac{2}{3} a \underbrace{\int_0^\infty \xi e^{-\xi}\, d\xi}_{1} = \underline{\underline{\frac{2}{3} a}}$

$$\langle x^2 \rangle = \int_{-\infty}^{\infty} x^2 p(x)\,dx = \frac{2}{3} a^2 \underbrace{\int_0^\infty \xi^2 e^{-\xi}\, d\xi}_{2} = \frac{4}{3} a^2$$

$$\text{Var}(x) = \langle x^2 \rangle - \langle x \rangle^2 = \left(\frac{4}{3} - \frac{4}{9}\right) a^2 = \underline{\underline{\frac{8}{9} a^2}}$$

d/ $\underline{\underline{p(d) \approx \frac{1}{\sqrt{2\pi\sigma^2}}\, e^{-(d-\langle d \rangle)^2/2\sigma^2}}}$ by CLT

with $\langle d \rangle = 36 \times \langle x \rangle = \underline{\underline{24 a}}$

$\sigma^2 = 36 \times \text{Var}(x) = \underline{\underline{32 a^2}}$

2. a/ $p(A) = \int_{-\infty}^{\infty} p(A,B)\,dB$ $\quad = \underline{\underline{0 \text{ if } A<0}}$

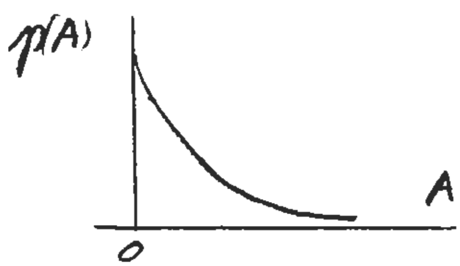
$$= \frac{\gamma^2}{\alpha^6}\int_0^{\infty} B^2 e^{-B(A+\gamma)/\alpha^2}\,dB$$

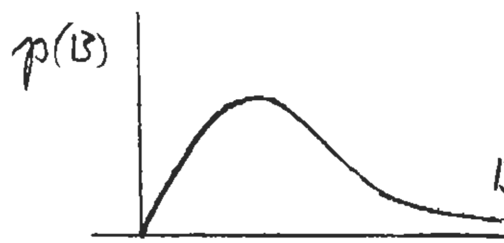
$$= \frac{\gamma^2}{\alpha^6}\left[\frac{\alpha^2}{(A+\gamma)}\right]^3 \underbrace{\int_0^{\infty} \xi^2 e^{-\xi}\,d\xi}_{2} = \underline{\underline{\frac{2\gamma^2}{(A+\gamma)^3}}} \quad A>0$$

$p(B) = \int_{-\infty}^{\infty} p(A,B)\,dA$ $\quad \underline{\underline{=0 \text{ if } B<0}}$

$$= \frac{B^2\gamma^2}{\alpha^6} e^{-B\gamma/\alpha^2} \underbrace{\int_0^{\infty} e^{-BA/\alpha^2}\,dA}_{\frac{\alpha^2}{B}\underbrace{\int_0^{\infty} e^{-\xi}\,d\xi}_{1}}$$

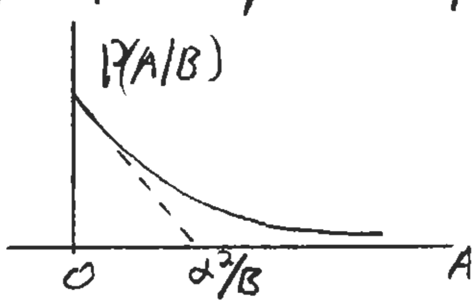
$$\underline{\underline{= \left(\frac{\gamma}{\alpha^2}\right)\left(\frac{B\gamma}{\alpha^2}\right) e^{-B\gamma/\alpha^2} \quad \text{if } B \geq 0}}$$





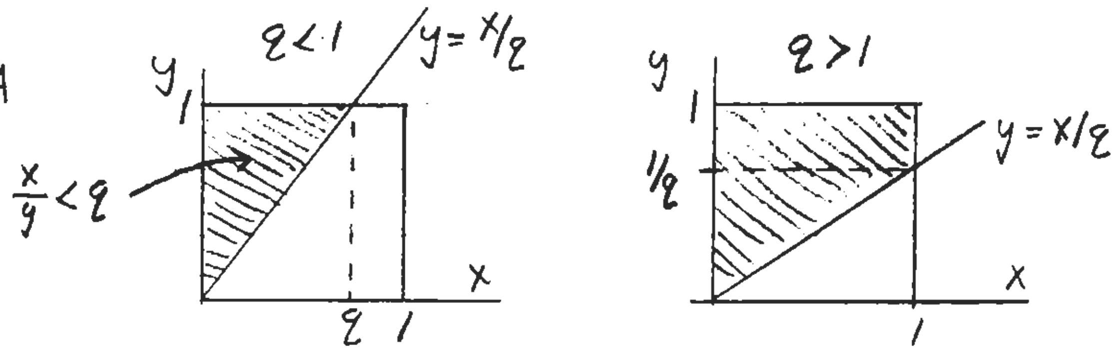
b/ $p(A|B) = p(A,B)/p(B) = \underline{\underline{\frac{B}{\alpha^2} e^{-A(B/\alpha^2)}}} \quad A \geq 0$

$= 0 \quad A<0$



c/ <u>A & B are not S.I. because $p(A,B) \neq p(A)\,p(B)$</u>

3. Step A



Note: geometry differs for $q<1$ and $q>1$

Step B $\quad p(x,y)=1$ in square, $=0$ outside

$q<1 \quad P(q) = \text{area of shaded region} = \frac{1}{2}q$

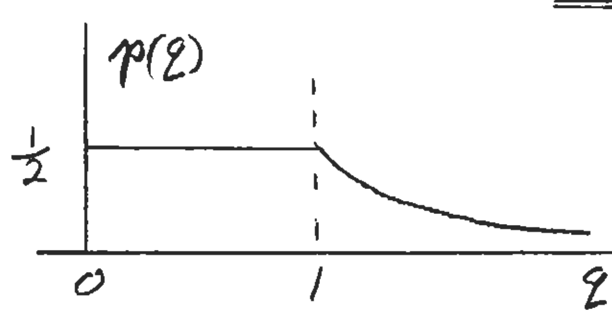
$q>1 \quad P(q) = \text{area of shaded region}$

$\quad = 1 - (\text{unshaded area in square})$

$\quad = 1 - \frac{1}{2}(1/q)$

Step C $\quad p(q) = \frac{d}{dq}P(q) = \underline{\underline{\frac{1}{2} \quad 0<q<1}}$

$\quad = \underline{\underline{\frac{1}{2}\left(\frac{1}{q^2}\right) \quad q>1}}$



$$\int_0^\infty p(q)\,dq = \frac{1}{2} + \int_1^\infty \frac{1}{2}\frac{dq}{q^2} = \frac{1}{2} + \frac{1}{2}\left[-\frac{1}{q}\right]_1^\infty = \frac{1}{2}+\frac{1}{2} = \underline{\underline{1}}$$