PROFESSOR: Last time we talked about particle on a circle. Today the whole lecture is going to be developed to solving Schrodinger's equation. This is very important, has lots of applications, and begins to give you the insight that you need to the solutions. So we're going to be solving this equation all through this lecture.

And let me remind you what with had with a particle on a circle. The circle is segment 0 to L, with L and 0 identified. More properly we actually think of the whole x-axis with the identification that two points related in this way that differ by L, or therefore for 2L, 3L, are the same point.

As a result, we want wave functions that have this periodicity. And that implies the same periodicity for the derivatives as well. We looked at the Schrodinger equation and we proved that the energy of any solution has to be positive or 0. And therefore the differential equation, the Schrodinger differential equation, can be read then as minus k squared psi, where k squared is this quantity and it's positive, so k is a real number. That makes sense.

And finally, once you have this equation, you know that if the second derivative of a function is proportional to minus the function, the solution are trigonometric functions or exponentials. And we decided to go for exponentials, that they are perhaps a little more understandable, though we will go back to them.

Now that's where we stopped last time. And now we apply the periodicity condition. So we must have e to the ik x plus L equal to e to the ikx. If you cancel the e to the ikx on both sides, you get to e to the ikL must be equal to 1, which forces kL to be a multiple of pi. That is, kL equal 2 pi n-- of 2 pi, I'm sorry-- 2 pi n, where n is an integer.

So those are the values of k. We'll write them slightly differently. We'll write kn with the subscript n to represent the k determined by the integer n. So it will be 2 pi n over L. Now from that equation, for k squared equal 2mE over h squared, you get e is equal to h squared k squared over 2m. And k in these solutions represents therefore the momentum. That is, the momentum Pn is h bar kn, and it's 2 pi h bar n over L. And the energies associated with solutions with kn value is En would be h squared kn squared, so 4 pi squared n squared over L squared over 2m. So this is equal to 2 pi squared h squared n squared over m L squared.

Those are numbers. It's good to have them. Our solution is psi n of x is equal to e to the i, or is proportional to, e to the i knx. So far so good.

But we can now normalize this thing. This is the beauty of the problem of a particle on a circle. If you have a particle in free space, psi squared is equal to 1 and the integral is infinite. On the other hand, these ones are normalizable. That is, we can demand that the infinite over the circle of psi n squared be equal to 1.

So how do we do that? Well I'll write it a little more explicitly here. Psi n of x will be some constant times e to the iknx. And therefore this thing is the integral from 0 to L dx of the constant squared. The constant can be chosen to be real. N squared times psi n squared, which is-- this exponential squared is just 1. This is 1. So it just gives you L times N squared is equal to 1. So N is equal to 1 over square root of L.

And finally, our psi n's of x are 1 over square root of L e to the iknx or 1 over square root of L e to the 2 pi inx over L. Oops. All right.

So these are our wave functions. These are our energy eigenstates. Our full stationary states, where we're finding stationary states-- stationary states have a psi of x times a time dependence. The time dependence is e to the minus i e-- so you could say that psi n of x and t, the full stationary state, is psi n of x times e to the minus i e n t over h bar. And that solves the full Schrodinger equation. That's our stationary state.

So one thing that should be emphasized here is the range of the integers. n is an integer and we better realize if there are some exceptions. Maybe just the positive? Is 0 included? Is 0 not included? And here, it's really as stated here. It's all the integers, n from minus infinity to plus infinity. All of them must be included.

The reason we can understand that is that the momentum of each of these states, the momentum is 2 pi h n over L. And therefore these are all states of different momentum. There's no question that these are different states. It cannot be that one is just the same as another one. They have different momentum. They represent the particle going with some momentum around the circle, and that momentum is quantified by n and it could be in the positive direction or negative direction.

Now you could be suspicious about n equals 0. But there's actually nothing to be suspicious about it. It's surprising. But psi 0 is 1 over square root of L, has no x dependence. And therefore it has 0 energy. And that's-- I'm sorry, here. There's some psi missing. The second derivative of a constant is 0. And if e is equal to 0, that's a consistent solution. The constant is important.

And now you also realize that psi, you have a nice phenomenon, that psi minus 1 and psi 1, for example, they correspond to n equals 1 and minus 1, have the same energy. Because energy depends on n squared, so these are degenerate states with energy E1 equal to E minus 1. And so are psi 2 and psi minus 2. And of course just psi minus k and psi k. They are degenerate states.

And now this hits into a property that is going to be important in the future about degenerate states. Whenever somebody gives you a couple of degenerate states, you know they have the same energy. But you must not stop there. If they are degenerate states and there are two states, it means that they are not the same. So there must be something physical about them that distinguishes them. Whenever you have degenerate states, you have to work until you figure out what is different about one state and the other.

And here we got the answer. The answer is simply that they are degenerate states with a different momentum. So the momentum is an observable that distinguishes those degenerate states. In fact, as we've written here, p on psi n of x is equal to Pn psi n. And P n given by this quantity. OK.

Our eigenstates are orthonormal. They're eigenstates. So why are they orthonormal? They are eigenstates of a Hermitian operator with different eigenvalues. They're eigenstates of p with different eigenvalues. So they're orthonormal.

The argument with the energy would have not worked out so well because there you have degenerate states. So these two states are degenerate with respect to energy. So you could wonder, how do you know they are orthogonal? But in this case it's simple. They have different momentum. Momentum is a Hermitian operator, and it should be orthogonal.

So the states are orthogonal. They are complete. You could write any wave function of the circle as a superposition of those psi n's. So any psi of x periodic can be written as psi of x the sum a n psi n's over all the integers .

And one last remark. We could have worked with sines and cosines. And therefore we could have worked with psi k plus psi minus k. This psi k and psi minus k have the same energy. Therefore this sum is an energy eigenstate of that same energy. The Hamiltonian acting on psi k gives you the energy times psi k. Here, the same energy times psi minus k, so this is an energy eigenstate. And this is proportional to cosine of kx.

And this is an energy eigenstate you know, because two derivatives of a cosine will give you back that cosine. Similarly, psi k minus psi of minus k is proportional to sine of kx. And that's also an energy eigenstate. Both are energy eigenstates.

So this is kind of the way you can reformulate Fourier's theorem here. You could say anything can be written as a superposition of all the exponentials, including the exponential with n equals 0, which is just a constant. Or alternatively, everything could be written in terms of sines and cosines, which is another way of doing the Fourier theorem.

These are energy eigenstates, but they're not P eigenstates anymore. This, when you take a derivative, becomes a sine. When this, you take a derivative, becomes a cosine. They're not energy. They're not momentum eigenstates.

So you can work with momentum eigenstates, you can work with energy eigenstates. It's your choice. It's probably easier to work just with momentum, I can say. So that's it for the particle in a circle. We have three problems to solve today. Particle in a circle, particle in a box, and particle in a finite well.