

PROFESSOR: $\hbar \frac{d\psi}{dt} = E\psi$ where E hat is equal to p^2 over $2m$, the operator. That is the Schrodinger equation. The free particle Schrodinger equation-- you should realize it's the same thing as this. Because p is \hbar over i ddx .

And now Schrodinger did the kind of obvious thing to do. He said, well, suppose I have a particle moving in a potential, a potential V of x and t -- potential. Then the total energy is kinetic energy plus potential energy.

So how about if we think of the total energy operator. And here is a guess. We'll put the just p^2 over $2m$, what we had before. That's the kinetic energy of a particle. But now add plus V of x and t , the potential.

That is reasonable from your classical intuition. The total energy is the sum of them. But it's going to change the Schrodinger equation quite substantially.

Now, most people, instead of calling this the energy operator, which is a good name, have decided to call this the Hamiltonian. So that's the most popular name for this thing. This is called the Hamiltonian H .

And in classical mechanics, the Hamiltonian represents the energy expressed in terms of position and momenta. That's what the Hamiltonian is, and that's roughly what we have here. The energy is [\hbar in \hbar] [\hbar terms \hbar] [\hbar of \hbar] momenta and position.

And we're going to soon be getting to the position operator, therefore. So this is going to be the Hamiltonian. And we'll put the hat as well.

So Schrodinger's inspiration is to say, well, this is going to be H hat. And I'm going to say that $\hbar \frac{d\psi}{dt} = H\psi$. Or equivalently, $\hbar \frac{d}{dt} \psi(x, t) = -\frac{\hbar^2}{2m} \frac{d^2\psi}{dx^2} - V(x, t)\psi$ -- that's the p^2 over $2m$ -- plus V of x and t , all multiplying ψ .

This is it. This is the full Schrodinger equation. So it's a very simple departure. You see, when you discover the show that the equation for a free particle, adding the energy was not that difficult. Adding the potential energy was OK.

We just have to interpret this. And maybe it sounds to you a little surprising that you multiply

this by ψ . But that's the only way it could be to be a linear equation. It cannot be that ψ is acted by this derivative, but then you add v .

It would not be a linear equation. And we've realize that the structure of the Schrodinger equation is $d\psi/dt$ is equal to an energy operator times ψ .

The whole game of quantum mechanics is inventing energy operators, and then solving these equations, then see what they are. So in particular, you could invent a potential and find the equations. And, you see, it looks funny. You've made a very simple generalization. And now you have an equation.

And now you can put the potential for the hydrogen atom and calculate, and see if it works. And it does. So it's rather unbelievable how very simple generalizations suddenly produce an equation that has the full spectrum of the hydrogen atom. It has square wells, barrier penetration, everything. All kinds of dynamics is in that equation.

So we're going to say a few more things about this equation now. And I want you to understand that the V , at this moment, can be thought as an operator. This is an operator, acts on a wave function to give you a function.

This is a simpler operator. It's a function of x and t . And multiplying by a function of x and t gives you a function of x and t . So it is an operator. Multiplying by a given function is an operator. It changes all the functions.

But it's a very simple one. And that's OK, but V of x and t should be thought as an operator. So, in fact, numbers can be operator. Multiplication by a number is an operator. It adds on every function and multiplies it by a number, so it's also an operator.

But x has showed up. So it's a good time to try to figure out what x has to do with these things. So that's what we're going to do now. Let's see what's x have to do with things.

OK, so functions of x , V of x and t multiplied by wave functions, and you think of it as an operator. So let's make this formal. Introduce an operator, \hat{X} , which, acting on functions of x , multiplies them by x .

So the idea is that if you have the operator \hat{X} acting on the function f of x , it gives you another function, which is the function x times f of x -- multiplies by x . And you say, wow, well, why do you have to be so careful in writing something so obvious? Well, it's a good idea to do

that, because otherwise you may not quite realize there's something very interesting happening with momentum and position at the same time, as we will discover now.

So we have already found some operators. We have operators P , x , Hamiltonian, which is p^2 over $2m$. And now you could put V of x hat t . You know, if here you put V of x hat, anyway, whatever x hat does is multiplied by x .

So putting V of x hat here-- you may want to do it, but it's optional. I think we all know what we mean by this. We're just multiplying by a function of x .

Now when you have operators, operators act on wave functions and give you things. And we mentioned that operators are associated or analogs of matrices. And there's one fundamental property of matrices. The order in which you multiply them makes a difference.

So we've introduced two operators, p and x . And we could ask whether the order of multiplication matters or not. And this is the way Heisenberg was lead to quantum mechanics. Schrodinger wrote the wave equation. Heisenberg looked at operators and commutation relations between them. And it's another way of thinking of quantum mechanics that we'll use.

So I want to ask the question, that if you have p and x and you have two operators acting on a wave function, does the order matter, or it doesn't matter? We need to know that. This is the basic relation between p and x .

So what is the question? The question is, if I have-- I'll show it like that-- x and p acting on a wave function, ϕ , minus px acting on a wave function, do I get 0? Do I get the same result or not?

This is our question. We need to understand these two operators and see how they are related. So this is a very good question. So let's do that computation.

It's, again, one of those computation that is straightforward. But you have to be careful, because at every stage, you have to know very well what you're doing. So if you have two operators like a and b acting on a function, the meaning of this is that you have a acting on what b acting on ϕ gives you.

That's what it means to have two things acting. Your first act with the thing on the right. You then act on the other one.

So let's look at this thing-- $x\hat{p} - p\hat{x}$ phi. So for the first one, you would have x times p hat on phi minus p hat times x on phi, phi of x and t maybe-- phi of x and t .

OK, now what do we have? We have x hat acting on this. And this thing, we already know what it is-- \hbar/i ddx of phi of x and t -- minus p hat and x , acting on phi, is little x phi of x and t .

Now this is already a function of x and t . So an x on it will multiply it by x . So this will be \hbar/i x ddx of phi. It just multiplies it by x at this moment-- minus here we have \hbar bar over i ddx of x phi.

And now you see that when this derivative acts on phi, you get a term that cancels this. But when it acts on x , it gives you an extra term. ddx of x is minus \hbar over i phi-- or $i\hbar$ phi.

So the derivative acts on x or an phi. When it acts on phi, gives you this term. When it acts on x , gives you the thing that is left over.

So actually, let me write this in a more clear way. If you have an operator, a linear operator A plus B acting on a function phi, that's A phi plus B phi. You have linear operators like that.

And we have these things here. So this is actually equal to x hat p hat minus p hat x hat on phi. That's what it means when you have operators here.

So look what we got, a very surprising thing. $x\hat{p} - p\hat{x}$ is an operator. It wants to act on function. So we put a function here to evaluate it. And that was good.

And when we evaluate it, we got a number times this function. So I could say-- I could forget about the phi. I'm simply right that $x\hat{p} - p\hat{x}$ is equal to $i\hbar$ bar.

And although it looks a little funny, it's perfectly correct. This is an equality between operators-- equality between operators. On the left-hand side, it's clear that it's an operator. On the right-hand side, it's also an operator, because a number acts as an operator on any function it multiplies by it.

So look what you've discovered, this commutator. And that's a notation that we're going to use throughout this semester, the notation of the commutator. Let's introduce it here.

So if you have two operators, linear operators, we define the commutator to be the product in the first direction minus the product in the other direction. This is called the commutator of A and B .

So it's an operator, again, but it shows you how they are non-trivial, one with respect to the other. This is the basis, eventually, of the uncertainty principle. x and p having a commutator of this type leads to the uncertainty principle.

So what did we learn? We learned this rather famous result, that the commutator of x and p in quantum mechanics is $i\hbar$.