PROFESSOR: What we want to understand now is really about momentum space. So we can ask the following question-- what happens to the normalization condition that we have for the wave function when we think in momentum variables?

So yes, I will do this first. So let's think of integral dx psi of x star psi of x. Well, this is what we called the integral, the total integral for x squared, the thing that should be equal to 1 if you have a probability interpretation, for the wave function. And what we would like to understand is what does it say about phi of k?

So for that, I have to substitute what [? size ?] r in terms of k and try to rethink about this integral how to evaluate it. So for example, here, I can make a little note that I'm going to use a variable of integration that I call k for the first factor. And for this factor, I'm going to use k prime.

You should use different variables of integration. Remember, psi of x is an integral over k. But we should use different ones not to get confused.

So here we go-- dx 1 over square root of 2 pi-- integral-- we said this one is over k. So it would be phi star-- let me put the dk first-- dk phi star of k e to the minus ikx and dk. That's the first psi. This is psi star.

And now we put psi. So this is the dk prime-- phi of x dk prime phi of k prime e to the ik prime x. It's the same x in the three places.

OK, at this moment, you always have to think, what do I do next? There are all these many integrals. Well, the integrals over k, there's no chance you're going to be able to do them apparently-- not to begin with, because they are abstract integrals. So k integrals have no chance. Maybe the integral that we have here, the dx, does have a chance.

So in fact, let me write this as integral dk phi star of k integral dk prime phi of k prime. And then I have 1 over 2 pi integral dx e to the ik prime minus kx. I think I didn't miss any factor.

And now comes to help this integral representation of the delta function. And it's a little opposite between the role of k and x. Here the integration variable is over x. There is was over k.

But the spirit of the equality or the representation is valid. You have the 1 over 2 pi, a full integral over a variable, and some quantity here. And this is delta of k prime minus k.

And finally, I can do the last integral. I can do the integral, say, over k prime. And that will just give me-- because a delta function, that's what it does. It evaluates the integrand at the value. So you integrate over k prime-- evaluates phi at k. So this is equal to integral dk phi star of k phi of k.

And that's pretty neat. Look what we found. We've found what is called Parseval's theorem, which is that integral dx of psi of x squared is actually equal to integral dk phi of k squared. So it's called Parseval theorem-- Parseval's theorem.

Sometimes in the literature, it's also called Plancherel's theorem. I think it depends on the generality of the identity-- so Plancherel's theorem.

But this is very nice for us, because it begins to tell us there's, yes indeed, some more physics to phi of k. Why? The fact that this integral is equal to 1 was a key thing.

Well, the fact that it didn't change in time thanks to showing the equation was very important. It's equal to 1. We ended up with a probabilistic interpretation for the wave function. We could argue that this could be a probability, because it made sense.

And now we have a very similar relation for phi of k. Not only phi of k represents as much physics as psi of k, as psi of x, and it not only represents the weight with which you superimpose plane waves, but now it also satisfies a normalization condition that says that the integral is also equal to this integral, which is equal to 1. It's starting to lead to the idea that this phi of k could be thought maybe as a probability distribution in this new space, in momentum space.

Now I want to make momentum space a little bit more clear. And this involves a little bit of moving around with constants, but it's important. We've been using k all the time. And momentum is h bar k.

But now let's put things in terms of momentum. Let's do everything with momentum itself. So let's put it here. So let's go to momentum space, so to momentum language-- to momentum language.

And this is not difficult. We have p is equal to h bar k. So dp over h bar is equal to dk in our integrals. And we can think of functions of k, but these are just other functions of momentum.

So I can make these replacements in my Fourier relations.

So these are the two equations that we wrote there-- are the ones we're aiming to write in a more momentum language rather than k, even though it's going to cost a few h bars here and there. So the first equation becomes psi of x equal 1 over square root of 2 pi integral.

Well, phi of k is now phi of p with a tilde. e to the ikx is e to the ipx over h bar. And dk is equal to dp over h bar.

Similarly, for the second equation, instead of phi of k, you will put phi tilde of p 1 over square root of 2 pi integral psi of x e to the minus ipx over h bar. And that integral doesn't change much [? dx. ?]

So I did my change of variables. And things are not completely symmetric if you look at them. Here, they were beautifully symmetric. In here, you have 1 over h bar here and no h bar floating around.

So we're going to do one more little change for more symmetry. We're going to redefine. Let phi tilde of p be replaced by phi of p times square root of h bar.

You see, I'm doing this a little fast. But the idea is that this is a function I invented. I can just call it a little different, change its normalization to make it look good. You can put whatever you want.

And one thing I did, I decided that I don't want to carry all these tildes all the time. So I'm going to replace phi tilde of p by this phi of p. And that shouldn't be confused with the phi of k. It's not necessarily the same thing, but it's simpler notation.

So if I do that here, look-- you will have a phi of p, no tilde, and 1 over square root of h bar. Because there will be e square root of h bar in the numerator and h bar there. So this first equation will become psi of x 1 over square root of 2 pi h bar integral phi of p e to the ipx over h bar dp.

And the second equation, here, you must replace it by a phi and a square root of h bar, which will go down to the same position here so that the inverse equation is phi of p, now 1 over square root of 2 pi h bar integral psi of x e to the minus ipx over h bar dx.

So this is Fourier's theorem in momentum notation, in which you're really something over momenta. And you put all these h bars in the right place. And we've put them symmetrically. You could do otherwise. It's a choice.

But look at the evolution of things. We've started with a standard theorem with k and x, then derived a representation for the delta function, derived Parseval's theorem, and finally, rewrote this in true momentum language.

Now you can ask what happens to Parseval's theorem. Well, you have to keep track of the normalizations what will happen. Look, let me say it.

This left-hand side, when we do all these changes, doesn't change at all. The second one, dk, gets a dp over h bar. And this is becomes phi tilde of p.

But phi tilde of p then becomes a square root, then it's phi of p. So you get two square roots in the numerator and the dk that had a h bar in the denominator. So they all disappear, happily. It's a good thing.

So Parseval now reads, integral dx psi squared of x equal integral dk phi of k-- phi of p, I'm sorry. I just doing p now. And it's a neat formula that we can use.