BARTON ZWIEBACH: We were faced last time with a question of interpretation of the Schrodinger wave function. And so to recap the main ideas that we were looking at, we derive this Schrodinger equation, basically derived it from simple ideas-- having operators, energy operator, momentum operator, and exploring how the de Broglie wavelength associated to a particle would be a wave that would solve the equation. And the equation was the Schrodinger equation, a free Schrodinger equation, and then we added the potential to make it interacting and that way, we motivated the Schrodinger equation and took this form of x and t psi of x and t. And this is a dynamical equation that governs the wave function.

But the interpretation that we've had for the wave function, we discussed what Born said, was that it's related to probabilities and psi squared multiplied by a little dx would give you the probability to find the particle in that little dx at some particular time. So psi of x and t squared dx would be the probability to find the particle at that interval dx around x.

And if you're describing the physics of your Schrodinger equation is that of a single particle, which is the case here-- one coordinate, the coordinate of the particle, this integral, if you integrate this all over space, must be 1 for the probability to make sense. So the total probability of finding the particle must be 1, must be somewhere. If it's in one part of another part or another part, this probabilities-- for this to be a probability distribution, it has to be well-normalized, which means 1.

And we said that this equation was interesting but somewhat worrisome, because if the normalization of the wave function satisfies, if this holds for t equal t-nought then the Schrodinger equation, if you know the wave function all over space for t equal t-nought, which is what you would need to know in order to check that this is working, a t equal t-nought, you take the psi of x and t-nought, integrate it. But if you know psi of x and t-nought for all x, then the Schrodinger equation tells you what the wave function is at a later time. Because it gives you the time derivative of the wave function in terms of data about the wave function all over space.

So automatically, the Schrodinger equation must make it true that this will hold at later times. You cannot force the wave function to satisfy this at all times. You can force it maybe to satisfy at one time, but once it satisfies it at this time, then it will evolve, and it better be that at every time later, it still satisfies this equation.

So this is a very important constraint. So we'll basically develop this throughout the lecture today. We're going to make a big point of this trying to explain why the conditions that we're going to impose on the wave function are necessary; what it teaches you about the Hamiltonian, we'll teach you that it's a Hermitian operator; what do you learn about probability-- you will learn that there is a probability current; and all kinds of things will come out of taking seriously the interpretation of this probability, the main point being that we can be sure it behaves as a probability at one time, but then for later times, the behaviors and probability the Schrodinger equation must help-- must somehow be part of the reason this works out. So that's what we're going to try to do.

Now, when we write an equation like this, and more explicitly, this means integral of psi star of x and t, psi of x and t dx equal 1. You can imagine that not all kind of functions will satisfy it. In particular, any wave function, for example, that at infinity approaches a constant will never satisfy this, because if infinity, you approach a constant, then the integral is going to be infinite. And it's just not going to work out. So the wave function cannot approach a finite number, a finite constant as x goes to infinity. So in order for this to hold-- order to guarantee this can even hold, can conceivably hold, it will require a bit of boundary conditions. And we'll say that the limit as x goes to infinity or minus infinity-- plus/minus infinity of psi of x and t will be equal to 0. It better be true. And we'll ask a little more.

Now, you could say, look, certainly the limit of this function could not be in number, because it would be non-zero number, the interval will diverge. But maybe there is no limit. The wave function is so crazy that it can be integrated, but suddenly, it has a little spike and it just doesn't have a normal limit. That could conceivably be the case. Nevertheless, it doesn't seem to happen in any example that is of relevance. So we will assume that the situations are not that crazy that this happened.

So we'll take wave functions that necessarily go to 0 at infinity. And that certainly is good. You cannot prove it's a necessary condition, but if it holds, it simplifies many, many things, and essentially, if the wave function is good enough to have a limit, then the limit must be 0.

The other thing that we will want is that d psi/dx, the limit as x goes to plus/minus infinity is bounded. That is, yes, the limit may exist and it may be a number, but it's not infinite. And In every example that I know of-- in fact, when this goes to 0, this goes to 0 as well-- but this is basically all you will ever need in order to make sense of the wave functions and their integrals that we're going to be doing.

Now you shouldn't be too surprised that you need to say something about this wave function in the analysis that will follow, because the derivative-- you have the function and its derivative, because certainly, there are two derivatives here. So when we manipulate these quantities inside the integrals, you will see very soon-- single derivatives will show up and we'll have to control them.

So the only thing that I'm saying is that when you see a wave function that satisfies this property, you know that unless the function is extremely crazy, it's a function that goes to 0 at plus/minus infinity. And it's the relative pursuant it also goes to 0, but it will be enough to say that it maybe goes to a number.

Now there's another possibility thing for confusion here with things that we've been saying before. We've said before that the physics of a wave function is not altered by multiplying the wave function by a number. We said that psi added to psi is the same state; psi is the same state as square root of 2 psi-- all this is the same physics, but here it looks a little surprising if you wish, because if I have a psi and I got this already working out, if I multiply psi by square root of 2, it will not hold. So there seems to be a little maybe something with the words that we're been using. It's not exactly right and I want to make sure there is no room for confusion here, and it's the following fact.

Here, this wave function has been normalized. So there's two kinds of wave functions that you can have-- wave functions that can be normalized and wave functions that cannot be normalized. Suppose somebody comes to you and gives you a psi of x and t. Or let's assume that-- I'll put x and t. No problem. Now suppose you go and start doing this integral-- integral of psi squared dx. And then you find that it's not equal to 1 but is equal to some value N, which is different from 1 maybe. If this happens, we say that psi is normalizable, which means it can be normalized.

And using this idea that changing the value-- the coefficient of the function-- doesn't change too much, we simply say, use instead psi prime, which is equal to psi over square root of N. And look what a nice property this psi prime has. If you integrate psi prime squared, it would be equal-- because you have psi prime here is squared, it would be equal to the integral of PSI squared divided by the number N-- because there's two of them-- dx, and the number goes out and you have the integral of psi squared dx, but that integral was exactly N, so that's 1.

So if your wave function has a finite integral in this sense, a number that is less than infinity, then psi can be normalized. And if you're going to work with probabilities, you should use instead this wave function, which is the original wave function divided by a number. So they realize that, in some sense, you can delay all of this and you can always work with wave functions that are normalizable, but only when you're going to calculate your probabilities. You can take the trouble to actually normalize them and those are the ones you use in these formulas.

So the idea remains that we work flexibly with wave functions and multiply them by numbers and nothing changes as long as you realize that you cannot change the fact that the wave function is normalizable by multiplying it by any finite number, it will still be normalized. And if it's normalizable, it's equivalent to a normalized wave function. So those two words sound very similar, but they're a little different. One is normalizable, which means it has an integral of psi squared finite, and normalize is one that already has been adjusted to do this and can be used to define a probability distribution.

OK. So that, in a way of introduction to the problem that we have to do, our serious problem is indeed justifying that the time evolution doesn't mess up the normalization and how does it do that?