BARTON ZWIEBACH: Today's subject is momentum space. We're going to kind of discover the relevance of momentum space. We've been working with wave functions that tell you the probabilities for finding a particle in a given position and that's sometimes called coordinate space or position space representations of quantum mechanics, and we just talked about wave functions that tell you about probabilities to find a particle in a given position.

But as we've been seeing with momentum, there's a very intimate relation between momentum and position, and today we're going to develop the ideas that lead you to think about momentum space in a way that is quite complimentary to coordinate space. Then we will be able to talk about expectation values of operators and we're going to be moved some steps into what is called interpretation of quantum mechanics. So operators have expectations values in quantum mechanism-- they are defined in a particular way and that will be something we're going to be doing in the second part of the lecture.

In the final part of the lecture, we will consider the time dependence of those expectation values, which is again, the idea of dynamics-- if you want to understand how your system evolves in time, the expectation value-- the things that you measure-- may change in time and that's part of the physics of the problem, so this will tie into the Schrodinger equation.

So we're going to begin with momentum space. So we'll call this uncovering momentum space. And for much of what I will be talking about in the first half of the lecture, time will not be relevant-- so time will become relevant later. So I will be writing wave functions that don't show the time, but the time could be put there everywhere and it would make no difference whatsoever.

So you remember these Fourier transform statements that we had that a wave function of x we could put time, but they said let's suppress time. It's a superposition of plane waves. So there is a superposition of plane waves and we used it last time to evolve wave packets and things like that. And the other side of Fourier's theorem is that phi of k can be written by a pretty similar integral, in which you put psi of x, you change the sign in the exponential, and of course, integrate now over x.

Now this is the wave function you've always learned about first, and then this wave function, as you can see, is encoded by phi of k as well. If you know phi of k, you know psi of x. And so this phi of k has the same information in principle as psi of x-- it tells you everything that you need to know. We think of it as saying, well, phi of k has the same information as psi of x. And the other thing we've said about phi of k is that it's the weight with which you're superposing plane waves to reconstruct psi of x. The Fourier transform theorem is a representation of the wave function in terms of a superposition of plane waves, and here, it's the coefficient of the wave that accompanies each exponential. So phi of k is the weight of the plane waves in the superposition.

So one thing we want to do is to understand even deeper what phi of k can mean. And so in order to do that, we need a technical tool. Based on these equations, one can derive a way of representing this object that we call the delta function. Delta functions are pretty useful for manipulating objects and Fourier transforms, so we need them. So let's try to obtain what is called the delta function statement.

And this is done by trying to apply these two equations simultaneously. Like, you start with psi of x, it's written in terms of phi of k, but then what would happen if you would substitute the value of phi of k in here? What kind of equation you get? What you get is an equation for a delta function.

So you begin with psi of x over square root of 2 pi, and I'll write it dk e to the ikx, and now, I want to write phi of k. So I put phi of k-- 1 over square root of 2 pi integral psi, and now I have to be a little careful. In here in the second integral, x is a variable of integration, it's a dummy variable. It doesn't have any physical meaning, per se-- it disappears after the integration. Here, x represents a point where we're evaluating the wave function, so I just cannot copy the same formula here because it would be confusing, it should be written with a different x and x-prime. Because that x certainly has nothing to do with the x we're writing here.

So here it is, we've written now this integral. And let's rewrite it still differently. We'll write as integral dx-prime. We'll change the orders of integration with impunity. If you're trying to be very rigorous mathematically, this is something you worry about. In physics problems that we deal with, it doesn't make a difference. So here we have dx-prime, psi of x-prime, then a 1 over 2 pi integral dk e to the ik x minus x-prime.

So this is what the integral became. And you look at it and you say, well, here is psi of x, and it's equal to the integral over x-prime of psi of x-prime times some other function of x and x-prime. This is a function of x minus x-prime, or x and x-prime if you wish. It doesn't depend on k, k is integrated. And this function, if you recognize it, it's what we call a delta function. It's a function that, multiplied with an integral, evaluates the integrand at a particular point. So this is a delta function, delta of x-prime minus x.

That is a way these integrals then would work. That is, when you integrate over x-prime-- when you have x-prime minus something, then the whole integral is the integrand evaluated at the point where we say the delta function fires. So the consistency of these two equations means that for all intents and purposes, this strange integral is a representation of a delta function. So we will write it down. I can do one thing here, but-- it's kind of here you see the psi of x-prime minus x, but here you see x minus x-prime.

But that's sign, in fact, doesn't matter. It's not there. You can get rid of it. Because if in this integral, you can let K goes to minus k, and then the dk changes sign, the order of integration changes sign, and they cancel each other. And the effect is that you change the sign in the exponents, so if you let k goes to minus k, the integral just becomes 1 over 2 pi integral dk e to the ik x-prime minus x. So we'll say that this delta function is equal to this thing, exploiting this sign ambiguity that you can always have. So I'll write it again in the way most people write the formula, which is at this moment, switching x and x-prime. So you will write this delta of x minus x-prime is 1 over 2 pi integral from infinity to minus infinity dk e to the ik x minus x-prime.

So this is a pretty useful formula and we need it all the time that we do Fourier transforms as you will see very soon. It's a strange integral, though. If you have x equal to x-prime, this is 0 and you get infinity. So it's a function, it's sort of 0-- when x minus x-prime is different from 0, somehow all these waves superimpose to 0. But when x is equal to x-prime, it blows up. So it's does the right thing, it morally does the right thing, but it's a singular kind of expression and we therefore manipulate it with care and typically, we use it inside of integrals.

So it's a very nice formula, we're going to need it, and it brings here for the first time in our course, I guess, the delta functions. And this is something if should-- if you have not ever play without the functions, this may be something interesting to ask in recitation or you can try to prove, for example, just like we show that delta of minus x is the same as delta of x, the delta of a being a number times x is 1 over absolute value of a times delta of x. Those are two simple properties of delta functions and you could practice by just trying to prove them-- for example, use this integral representation to show them.