PROFESSOR: Today we begin with the harmonic oscillator. And before we get into the harmonic oscillator, I want to touch on a few concepts that have been mentioned in class and just elaborate on them. It is the issue of nodes, and how solutions look at, and why solutions have more and more nodes, why the ground state has no nodes. This kind of stuff.

So these are just a collection of remarks and an argument for you to understand a little more intuitively why these properties hold. So one first thing I want to mention is, if you have a Schrodinger equation for an energy eigenstate. Schrodinger equation for an energy eigenstate. You have an equation of the from minus h squared over 2m. d second dx squared psi of x plus v of x psi of x equal e times psi of x. Now the issue of this equation is that you're trying to solve for two things at the same time.

If you're looking at what we call bound states, now what is a bound state? A bound state is something that is not extended all that much. So a bound state will be a wave function that goes to 0 as the absolute value of x goes to infinity.

So it's a probability function that certainly doesn't extend all the way to infinity. It just collapses. It's normalizable. So these are bound states, and you're looking for bound states of this equation. And your difficulty is that you don't know psi, and you don't know E either.

So you have to solve a problem in which, if you were thinking oh this is just a plain differential equation, give me the value of E. We know the potential, just calculate it. That's not the way it works in quantum mechanics, because you need to have normalizable solutions. So at the end of the day, as will be very clear today, this E gets fixed. You cannot get arbitrary values of E's.

So I want to make a couple of remarks about this equation. Is that there's this thing that can't happen. Certainly, if v of x is a smooth potential, then if you observer that the wave function vanishes at some point, and the derivative of the wave function vanishes at that same point, these two things imply that psi of x is identically 0.

And therefore it means that you really are not interested in that. That's not a solution of the Schrodinger equation. Psi equals 0 is obviously solves this, but it's not interesting. It doesn't represent the particle.

So what I claim here is that, if it happens to be that you're solving the Schrodinger problem with some potential that is smooth, you can take derivatives of it. And then you encounter that the wave function vanishes at some point, and its slope vanishes at that same point. Then the wave function vanishes completely.

So you cannot have a wave function, a psi of x that does the following. Comes down here. It becomes an inflection point and goes down. This is not allowed. If the wave function vanishes at some point, then the wave function is going to do this. It's going to hit at an angle, because you cannot have that the wave function is 0 and needs the derivative 0 at the same point.

And the reason is simple. I'm not going to prove it now here. It is that you have a second order differential equation, and a second order differential equation is completely determined by knowing the function at the point and the the derivative at the point. And if both are 0s, like the most trivial kind of initial condition, the only solution consistent with this is psi equals 0 everywhere.

So this can't happen. And it's something good for you to remember. If you have to do a plot of a wave function, you should never have this. So this is what we call the node in a wave function. It's a place where the wave function vanishes, and the derivative of the wave function better not vanish at that point.

So this is one claim that it's not hard to prove, but we just don't try to do it. And there's another claim that I want you to be aware of. That for bound states in one dimension, in the kind of thing that we're doing now in one dimension, no degeneracy is possible.

What do I mean by that? You will never find two bound states of a potential that are different that have the same energy. It's just absolutely impossible. It's a very wonderful result, but also I'm not going to prove it here. Maybe it will be given as an exercise later in the course. And it's discussed in 805 as well. But that's another statement that is very important. There's no degeneracy.

Now you've looked at this simple potential, the square well infinite one. And how does it look? You have x going from 0 to a. And the potential is 0. From 0 to a is infinite otherwise. The particle is bound to stay inside the two walls of this potential.

So in here we've plotted the potential as a function of x. Here is x. And the wave functions are things that you know already. Yes?

AUDIENCE: Is it true if you have two wells next to each other that there's still no degeneracy if it's an infinite barrier?

PROFESSOR: If there's two wells and there's an infinite barrier between them, it's like having two universes. So it's not really a one dimensional problem. If you have an infinite barrier like two worlds that can talk to each other. So yes, then you would have degeneracy.

It's like saying you can have here one atom of hydrogen or something in one energy level, here another one. They don't talk to each other. They're degenerate states. But in general, we're talking about normal potentials that are preferably smooth. Most of these things are true even if they're not smooth. But it's a little more delicate. But certainly two potentials that are separated by an infinite barrier is not part of what we really want to consider.

OK so these wave functions start with m equals 0, 1, 2, 3, and psi ends of x are square root of 2 over a sin n plus 1 pi x over a. Things that you've seen already. And En, the energies are ever growing as a function of the integer n that characterizes them. 2ma squared.

And the thing that you notice is that psi 0 has technically no nodes. That is to say these wave functions have to varnish at the end, because the potential becomes infinite, which means a particle really can go through. The wave function has to be continuous. There cannot be any wave function to the left. So it has to vanish here.

These things don't count as nodes. It is like a bound state has to vanish at infinity. And that's not what we count the node. A node is somewhere in the middle of the range of x where the wave function vanishes.

So this is the ground state. This is psi zero has no nodes. Psi one would be something like that, has one node. And try the next ones. They have more and more nodes. So psi n has n nodes.

And the interesting thing is that this result actually is true for extremely general potentials. You don't have to just do the square well to see that the ground state has no nodes. That first excited state was one node, and so on and so forth. It's true in general.

This is actually a very nice result, but its difficult to prove. In fact, it's pretty hard to prove. So there is a nice argument. Not 100% rigorous, but thoroughly nice and really physical that I'm going to present to you why this result is true. So let's try to do that.

So here is the general case. So I'm going to take a smooth v of x. That will be of this kind. The potential, here is x, and this potential is going to be like that. Smooth all over. And I will actually have it that it actually goes to infinity as x goes to infinity. Many of these things are really not necessary, but it simplifies our life.

OK, so here is a result that it's known to be true. If this thing grows to infinity, the potential never stops growing, you get infinite number of bound states at fixed energies. One energy, two energy, three energy, infinite number of bound states. Number of bound states. That's a fact. I will not try to prove it. We'll do it for the harmonic oscillator. We'll see those infinite number of states, but here we can't prove it easily.

Nevertheless, what I want to argue for you is that these states, as the first one, will have no nodes. The second state, the first excited state, will have one node. Next will have two nodes, three nodes, four nodes. We want to understand what is this issue of the nodes. OK.

You're not trying to prove everything. But we're trying to prove or understand something that is very important. So how do we prove this? Or how do we understand that the nodes-- so there will be an infinite number of bound states, psi 0, psi 1, psi 2, up to psi n, and it goes on. And psi n has n nodes. All right.

So what I'm going to do, in order to understand this, is I'm going to produce what we will call screened potentials. Screened potentials. I'm going to select the lowest point of the potential here for convenience. And I'm going to call it x equals 0 is the lowest point.

And the screen potential will have a parameter a. It's a potential which is equal to v of x if the absolute value of x is less than a. And its infinity if the absolute value of x is greater than a.

So I come here, and I want to see this is this potential, v of x, what is the screened potential for sum a? Well, I wanted colored chalk, but I don't have it. I go mark a here minus a. Here are the points between x and a. Absolute value of x less than a.

Throughout this region the screened potential is the potential that you have. Nevertheless, for the rest, its infinite. So the screened potential is this thing. Is infinite there, and it's here this thing. So it's just some potential. You take it a screen and you just see one part of the potential, and let it go to infinity. So that's a screen potential.

So now what I'm going to do is that I'm going to try to argue that you could try to find the bound state of the screen potential. Unless you remove the screen, you will find, as you let a go to infinity, you will find the bound states of the original potential. It's reasonable that that's true, because as you remove the screen, you're letting more of the potential be exposed, and more of the potential be exposed. And the wave functions eventually die, so as the time that you're very far away, you affect the wave functions less and less.

So that's the argument. We're going to try to argue that we're going to look at the bound states of the screened potentials and see what happened, whether they tell us about the bound states the original potential.

So for this, I'm going to begin with a screen potential in which a goes to 0 and say that a is equal to epsilon, very small. So what potential so do I have? A very tiny potential here from epsilon to minus epsilon. Now I chose the original point down here to be the minimum.

So actually, the bottom part of the potential is really flat. And if you take epsilon going to 0, well, the potential might do this, but really at the bottom for sufficiently small epsilon, this is an infinite square well with psis to epsilon. I chose the minimum so that you don't get something like this. If it would be a point with a slope, this would be an ugly thing. So let's choose the minimum. And we have the screen potential here, and that's it.

Now look what we do. We say all right, here there is a ground state. Very tiny. Goes like that. Vanishes here. Vanishes there. And has no nodes. Very tiny. You know the two 0s are very close to each other. And now I'm going to try to increase the value of the screen a.

So suppose we've increased the screen, and now the potential is here. And now we have a finite screen. Here is the potential. And I look at the wave function. How it looks. Here is psi 0. This ground state psi 0.

Well, since this thing in here, the potential becomes infinite, the wave function still must vanish here and still must vanish here. Now just for your imagination, think of this. At this stage, it still more or less looks like this. Maybe. Now I'm going to ask, as I increase, can I produce a node? And look what's going to happen.

So suppose it might happen that, as you increase, suddenly you produce a node.

So here's what I'm saying here. I'm going to show it here. Suppose up to this point, there is no node. But then when I double it, when I increase it to twice the size, when I go to screen potential like that, suddenly there is a node in the middle.

So if there is a node in the middle, one thing that could have happened is that you have this. And now look what must have happened then. As I stretch this, this slope must have been going down, and down, and down, until it flips to the other side to produce a node here. It could have happened on this side, but it's the same, so the argument is just done with this side.

To produce a node you could have done somehow the slope here must have changed sine. But for that to happen continuously, at some point the this slope must have been 0. But you cannot have a 0 and 0 slope. So this thing can't flip, can't do this.

Another thing that could have happened is that when we are here already, maybe the wave function looks like that. It doesn't flip at the edges, but produces something like that. But the only way this can happen continuously, and this potential is changing continuously, is for this thing at some intermediate stage, as you keep stretching the screen, this sort of starts to produce a depression here. And at some point, to get here it has to do this. But it can't do this either. It cannot vanish and have derivative like that.

So actually, as you stretch the screen, there's no way to produce a node. That property forbids it. So by the time you go and take the screen to infinity, this wave function has no nodes. So that proves it that the ground state has no nodes. You could call this a physicist proof, which means-- not in the pejorative way. It means that it's reasonable, it's intuitive, and a mathematician working hard could make it rigorous.

A bad physicist proof is one that is a little sloppy and no mathematician could fix it and make it work. So I think this is a good physics proof in that sense. Probably you can construct a real proof, or based on this, a very precise proof.

Now look at excited states. Suppose you take now here this screen very little, and now consider the third excited state, psi three. I'm sorry, we'll call this psi 2 because it has two nodes. Well, maybe I should do psi 1. Psi 1. One node.

Same thing. As you increase it, there's no way to create another node continuously. Because again, you have to flip at the edges, or you have to depress in the middle. So this one will evolve to a wave function that will have one node in the whole big potential.

Now stayed does that state have more energy than the ground state? Well, it certainly begins with a small screen with more energy, because in the square well psi 1 has more energy. And that energy should be clear that it's not going to go below the energy of the ground state.

Why? Because if it went below the energy of the ground state slowly, at some point for some value of the screen, it would have the same energy as the ground state. But no degeneracy is possible in one dimensional problems. So that can't happen. Cannot have that. So it will always stay a little higher. And therefore with one node you will be a little higher energy. With two nodes will be higher and higher. And that's it. That's the argument.

Now, we've argued by this continuous deformation process that this potential not only has these bound states, but this is n nodes and En is greater than En prime for n greater than n prime. So the more nodes, the more energy. Pretty nice result, and that's really all I wanted to say about this problem. Are there any questions? Any? OK.

So what we do now is the harmonic oscillator. That's going to keep us busy for the rest of today's lecture. It's a very interesting problem. And it's a most famous quantum mechanics problem in a sense, because it happens to be useful in many, many applications. If you have any potential-- so what is the characteristic of the harmonic oscillator? Harmonic oscillator. Oscillator. Well, the energy operator is p squared over 2m plus, we write, one half m omega squared x squared where omega is this omega that you always think of angular velocity, or angular frequency.

It's more like angular frequency. Omega has units of 1 over time. It's actually put 2pi over the period of an oscillation.

And this you know from classical mechanics. If you have a harmonic oscillator of this form, yeah, it actually oscillates with this frequency. And E is the energy operator, and this is the energy of the oscillator.

So what defines an oscillator? It's something in which the potential energy, this term is v of x. v of x is quadratic in x. That is a harmonic oscillator. Then you arrange the constants to make sense. This has units of energy, because this has units of length squared. 1 over time squared. Length over time is velocity squared times mass is kinetic energy. So this term has the units of energy. And you good with that.

And why is this useful? Because actually in any sort of arbitrary potential, rather general potential at least, whenever you have a minimum where the derivative vanishes, then the second derivative need not vanish. Then it's a good approximation to think of the potential at the minimum as a quadratic potential. It fits the potential nicely over a good region.

And therefore when you have two molecules with a bound or something oscillating, there is a potential. It has a minimum at the equilibrium position. And the oscillations are governed by some harmonic oscillator. When you have photons in space time traveling, there is a set of harmonic oscillators that correspond to photons. Many, many applications. Endless amount of applications for the harmonic oscillator. So we really want to understand this system quantum mechanically.

And what does that mean? Is that we really want to calculate and solve the Schrodinger equation. This is our first step in understanding the system. There's going to be a lot of work to be done even once we have the solutions of the Schrodinger equation. But the first thing is to figure out what are the energy eigenstates or the solutions of the Schrodinger equation for this problem.

So notice that here in this problem there's an energy quantity. Remember, when you have a harmontonian like that, and people say so what is the ground state energy? Well, have to find the ground state wave function. Have to do things. Give me an hour, I'll find it. And all that.

But if you want an approximate value, dimensional analysis will do it, roughly what is it going to be. Well, with this constant how do you produce an energy? Well, you remember what Einstein did, and you know that h bar omega has units of energy. So that's an energy associated with Lagrangian energy like quantity. And we expect that that energy is going to be the relevant energy. And in fact, we'll find that the ground state energy is just one half of that.

There's another quantity that may be interesting. How about the length? How do you construct a length from these quantities? Well, you can start doing m omega h bar and put powers and struggle. I hate doing that. I always try to find some way of doing it and avoiding that thing.

So I know that energies go like h over h squared over m length squared. So I'm going to call the length a quantity a. So ma squared. That has units of energy. And you should remember that because energy is b squared over 2m, and b by De Broglie is h bar over sub lamda. So h bar squared, lambda squared, and m here, that's units of energy. So that's a length.

On the other hand, we have another way to construct an energy is with this thing, m omega squared length squared. So that's also m omega squared a squared. That's another energy.

So from this equation I find that a to the fourth is h squared over m squared omega squared. And it's a little complicated, so a squared is h bar over m omega. So that's a length. Length squared. I don't want to take the square root. We can leave it for a moment there.

But that's important because of that's a length scale. And if somebody would ask you in the ground state, how far is this particle oscillating, you would say probably about a square root of this. Would be a natural answer and probably about right.

So OK, energy and units is very important to begin your analysis. So what is the

Schrodinger equation? The Schrodinger equation for this thing is going to be minus h squared over 2m, d second psi, dx squared plus the potential, one half m omega squared x squared psi is equal E psi. And the big problem is I don't know psi and I don't know E.

Now there's so many elegant ways of solving the harmonic oscillator. You will see those next lecture. Allan Adams will be back here. But we all have to go through once in your life through the direct, uninspired method of solving it. Because most of the times when you have a new problem, you will not come up with a beautiful, elegant method to avoid solving the differential equation. You will have to struggle with the differential equation. So today we struggle with the differential equation. We're going to just do it. And I'm going to do it slow enough and in detail enough that I hope you follow everything. I'll just keep a couple of things, but it will be one line computations that I will skip.

So this equation is some sort of fairly difficult thing. And it's complicated and made fairly unpleasant by the presence of all these constants. What kind of equation is that with all these constants? They shouldn't be there, all this constants, in fact.

So this is the first step, cleaning up the equation. We have to clean it up. Why? Because the nice functions in life like y double prime is equal to minus y have no units. The derivatives create no units. y has the same units of that, and the solution is sine of x, where x must have no units, because you cannot find the sine of one centimeter.

So this thing, we should have the same thing here. No units anywhere. So how can we do that? This is an absolutely necessary first step. If you're going to be carrying all these constants, you'll get nowhere. So we have to clean it up.

So what I'm going to try to see is that look, here is psi, psi, and psi. So suppose I do the following thing, that I will clean up the right hand side by dividing by something with units of energy. So I'm going to do the following way.

I'm going to divide all by 1 over h bar omega. And this 2 I'm going to multiply by 2.

So multiply by 2 over h bar omega. So what do I achieve with that first step? I achieve that these 2s disappear. Well, that's not too bad. Not that great either, I think. But in the right hand side, this has units of energy. And the right hand side will not have units of energy.

So what do we get here? So we get minus. The h becomes an h alone over-- the m disappears-- so m omega. The second psi the x squared. The 1/2 disappeared, so m omega over h bar x squared psi equals 2 E over h bar omega psi.

It looks actually quite better already. Should agree with that. It looks a lot nicer Now I can use a name for this. I want to call this the dimensionless value of the energy. So a calligraphic e. It has no units. It's telling me if I find some energy, that that energy really is this number, this pure number is how many times bigger is e with respect to h omega over 2.

So I'll write this now as e psi. And look what I have. I have no units here. And I have a psi. And I have a psi. But things have worked out already. Look, the same factor here, h over m omega is upside down here. And this factor has units of length squared. Length squared times d d length squared has no units. And here's 1 over length squared. 1 over length squared times length squared.

So things have worked out. And we can now simply say x is going to be equal to au, a new variable. This is going to be your new variable for your differential equation in which is this thing. And then this differential equation really has cleaned up perfectly well.

So how does it look now? Well, it's all gone actually, because if you have x equals au, d dx by chain rule is 1 over a d du. And to derivatives this with respect to x it's 1 over a squared times the d second du squared. And this thing is a squared. So actually you cancel this factor. And when I write x equals to au, you get an a squared times this. And a squared times this is 1.

So your differential equations has become minus the second psi du squared, where u is a dimensionless quantity, because this has units of length, this has units of length. No units here. You have no units. So minus d second du squared plus u squared psi is equal to e psi. Much nicer. This is an equation we can think about without being distracted by this endless amount of little trivialities.

But still we haven't solved it, and how are we going to solve this equation? So let's again think of what should happen. Somehow it should happen that these e's get fixed. And there is some solution just for some values of e's. It's not obvious at this stage how that is going to happen. Yes?

AUDIENCE: [INAUDIBLE].

PROFESSOR: Here for example, let me do this term. h bar over m omega is minus, from that equation, a squared. But dx squared is 1 over a squared d du squared. So a squared cancels. And here the x is equal a squared times u, so again cancels.

OK so what is the problem here? The problem is that most likely what is going to go wrong is that this solution for arbitrary values of e's is going to diverge at infinity, and you're never going to be able to normalize it. So let's try to understand how the solution looks as we go to infinity.

So this is the first thing you should do with an equation like that. How does this solution look as u goes to infinity? Now we may not be able to solve it exactly in that case either, but we're going to gain insight into what's happening.

So here it is. When u goes to infinity, this term, whatever psi is, this term is much bigger than that, because we're presumably working with some fixed energy that we still don't know what it is, but it's a fixed number and, for you, sufficiently large. This is going to dominate. So the equation that we're trying to solve as u goes to infinity, the equation sort of becomes psi double prime-- prime is for two derivatives-- is equal to u squared psi.

OK, so how do we get an idea what solves this is not all that obvious. It's certainly not a power of u, because when you differentiate the power of u, you lower the degree rather than increase the degree. So what function increases degree as you differentiate? It's not the trivial function. Cannot be a polynomial. If it could be even a polynomial, if you take two derivatives, it kind cannot be equal to x squared times a polynomial. It's sort of upside down.

So if you think about it for a little while, you don't have an exact solution, but you would imagine that something like this would do it, an e to the u squared. Because an e to the u squared, when you differentiate with respect to us, you produce a u down. When you one derivative. When you take another derivative, well, it's more complicated, but one term you will produce another u down. So that probably is quite good.

So let's try that. Let's try to see if we have something like that. So I will try something. I'll try psi equals 2. I'm going to try the following thing. e to the alpha u squared over 2 where alpha is a number. I don't know how much it is. Alpha is some number.

Now could try this alone, but I actually want to emphasize to you that if this is the behavior near infinity, it won't make any difference if you put here, for example, something like u to the power k. It will also be roughly a solution. So let's see that.

So for that I have to differentiate. And let's see what we get. So we're trying to see how the function behaves far, far away. You might say well look, probably that alpha should be negative. But let's see what the equation tells us before we put anything in there. So if I do psi prime, you would get what? You would get one term that would be alpha u times this u to the k into the alpha u squared over 2. I differentiated the exponential. I differentiated the exponential.

And then you would get a term where you differentiate the power. So you get ku to the k minus 1 into the alpha u squared over 2. If you take a second derivative, well, I can differentiate the exponential again, so I will get alpha u now squared, because each derivative of this exponent produces a factor of alpha u. u to the k into the alpha u squared over 2. And a couple more terms that they all have less powers of u, because this term has u to the k plus-- already has u to the k plus 1. And this has u to the k minus 1. They differ by two powers of u.

So for illustration, please, if you want, do it. Three lines, you should skip three lines in your notebook if you're taking notes and get the following. No point in me doing this algebra here. Alpha u squared over 2. Because actually it's not all that important. Over alpha 1 over u squared plus k minus 1 over alpha squared 1 over u to the fourth. That's all you get.

Look, this is alpha squared u squared times psi times these things. 1 plus 2 k plus 1 over alpha 1 over u squared. So when u goes to infinity, your solution works, because these thing's are negligible. So you get a number times u squared. That is the equation you are trying to solve up there. And therefore, you get that the equation if alpha squared is equal to 1. And that means and really that alpha can be plus minus 1. And roughly this solution near infinity, probably there's two solutions. This is a second order differential equation, so even near infinity there should be two solutions.

So we expect as u goes to infinity psi of u will be some constant A times u to the k times e to the minus u squared over 2. That's where alpha equal minus 1. Plus Bu to the k into the plus u squared over 2. And what is k? Well, we don't know what is k. It seems to work for all k. That may seem a little confusing now, but don't worry. We'll see other things happening here very soon.

So look at what has happened. We've identified that most likely your wave function is going to look like this at infinity. So we're going to want to this part not to be present. So presumably we're going to want a solution that just has this, because this is normalizable. The integral of any power times a Gaussian is convergence. So this can be normalized. The Gaussian falls so fast that any power can be integrated against a Gaussian. Any power however big doesn't grow big enough to compensate a Gaussian. It's impossible to compensate a Gaussian.

So we hope for this. But we want to translate what we've learned into some technical advantage in solving the differential equation, because, after all, we wanted be insight how it looks far way, but we wanted to solve the differential equation. So how can we use this insight we now have to simplify the solution of the differential equation?

The idea is to change variables a little bit. So write psi of u to be equal to h of u times e to the minus u squared over 2. Now you're going to say wait, what are you doing? Are you making an approximation now that this is what is going to look far away? Or what are you putting there? I'm not making any approximation.

I'm just saying whatever pis is, it can always be written in this way. Why? Because if you have a psi of u, you can write it as psi of u over e to the minus u squared over 2 times e minus u squared over 2. Very trivially this can always be done. As long as we say that h is arbitrary, there's nothing, no constraint here. I have not assume anything, nothing.

I'm just hoping that I have a differential equation for psi. That because this is a very clever factor, the differential equation for h will be simpler. Because part of the dependence has been taken over. So maybe h, for example, could be now a polynomial solution, because this product has been taken care.

So the hope is that by writing this equation it will become an equation for h of u, and that equation will be simpler. So will it be simpler? Well, here again this is not difficult. You're supposed to plug into equation one-- this is the equation one-- plug into one. I won't do it. It's three lines of algebra to plug into one and calculate the equation for h of u. You should do it. It's the kind of thing that one should do at least once. So please do it. It's three, four lines. It's not long. But I'll just write the answer.

So by the time you substitute, of course, the e to the minus u squared over 2 is going to cancel from everywhere. It's very here. You just need to take two derivatives, so it becomes a second order differential equation. And indeed, it becomes a tractable differential equation. The second h, du squared minus 2u dh du plus e minus 1 h equals 0.

OK, that is our equation now. So now we face the problem finally solving this equation. So before we start, maybe there's some questions of what we've done so far. Let's see. Any questions? Yes?

AUDIENCE: Do you have right there in the middle would be-- this equation is linear, so can we just [INAUDIBLE] minus u squared over 2 and you stuck it to that u to the k.

PROFESSOR: It's here? This thing?

AUDIENCE: Yeah. Could you then just power series what's going on at 0 with those u to the k terms [INAUDIBLE]?

PROFESSOR: No. This is the behavior as u goes to infinity. So I actually don't know that the function near 0 is going to behave like u to the k. We really don't know. It suggest to you that maybe the solution is going to be near 0 u to the k times some polynomial or something like that. But it's not that, because this analysis was just done at infinity. So we really have no information still what's going on near 0. Other questions? Yes?

AUDIENCE: So is k some arbitrary number or is it an integer?

PROFESSOR: At this moment, actually, it doesn't matter. Is that right? Doesn't matter. The analysis that we did here suggests it could be anything. That's why I just didn't put it into h or u. I didn't put it because would be strange to put here a u to the k. I wouldn't know what to make of it. So at this moment, the best thing to say is we don't know what it is, and maybe we'll understand it. And we will. In a few seconds, we'll sort of see what's going on.

OK, so how does one solve this equation? Well, it's not a trivial equation, again. But it can be solved by polynomials, and we'll see that. But the way we solve this equation is by a power series expansion. Now you could do it by hand first, and I did it when I was preparing the lecture yesterday. I said I'm going to just write h of u equal a constant a0 plus a1u plus a2u squared plus a3u cubed.

And I plugged it in here. And I just did the first few terms and start to see what happened. And I found after a little thinking that a2 is determined by a0, and a3 is determined by a1 once you substitute. It's not the obvious when you look at this, but that happens.

So when you face a problem like that, don't go high power to begin with. Just try a simple series and see what happens. And you see a little pattern. And then you can do a more sophisticated analysis. So what would be a more sophisticated analysis? To write h of u equal the sum from j equals 0 to infinity aju to the j. Then if you take a derivative, because we're going to need the derivative, dh du would be the sum from j equals 0 to infinity. j times aju to the j minus 1.

You would say that doesn't look very good because for j equals 0 you have 1 over u. That's crazy. But indeed for j equals 0, the j here multiplies it and makes it 0. So this is OK.

Now the term that we actually need is minus 2u dh du. So here minus 2u dh du would be equal to the sum from j equals 0 to infinity, and I will have minus 2jaju to the j. The u makes this j minus 1 j, and the constant went there.

So here is so far h. Here is this other term that we're going to need for the differential equation. And then there's the last term that we're going to need for the differential equation, so I'm going to go here. So what do we get for this last term. We'll have to take a second derivative. So we'll take-- h prime was there, so d second h du squared will be the sum from j equals 0 of j times j minus 1 aju to the j minus 2.

Now you have to rewrite this in order to make it tractable. You want everything to have u to the j's. You don't want actually to have u to the j minus 2. So the first thing that you notice is that this sum actually begins with 2, because for 0 and 1 it vanishes. So I can write j times j minus 1 aj u to j minus 2. Like that.

And then I can say let j be equal to j prime plus 2. Look, j begins with 2 in this sum. So if j is j prime plus 2, j prime will begin with 0. So we've shifted the sum so it's j prime equals 0 to infinity. And whenever I have a j I must put j prime plus 2. So j prime plus 2. j prime plus 1 aj prime plus 2 u to the j prime. Wherever I had j, I put j prime plus 2. And finally you say j or j prime is the same name, so let's call it j. j equals 0. j plus 2. j plus 1. aj plus 2 uj.

So we got the series expansion of everything, so we just plug into the differential equation. So where is the differential equation? It's here. So I'll plug it in. Let's see what we get. We'll get some from j equals 0 to infinity. Let's see the second derivative is here. j plus 2 times j plus 1 aj plus 2 uj, so I'll put it here. So that's this second derivative term.

Now this one. It's easy. Minus 2j aj and the uj is there. So minus 2jaj. Last term is just e minus 1, because it's the function this times aj as well. That's h. And look, this whole thing must be 0. So what you learn is that this coefficient must be 0 for every value of j.

Now it's possible to-- here is aj and aj, so it's actually one single thing. Let me write it here. j plus 2 times j plus 1 aj plus 2 minus 2j plus 1 minus e aj uj. I think I got it right. Yes. And this is the same sum.

And now, OK, it's a lot of work, but we're getting there. This must be 0. So actually that solves for aj plus 2 in terms of aj. What I had told you that you can notice in two minutes if you try it a little. That a2 seems to be determined by a0. And a3 seems to be determined by a2. So this is saying that aj plus 2 is given by 2j plus 1 minus e over j plus 2 j plus 1 aj. A very nice recursive relation.

So indeed, if you put the value of a0, it will determine for you a2, a4, a6, a8, all the even ones. If you put the value of a1, it will determine for you a3, a5. So a solution is determined by you telling me how much is a0, and telling me how much is a1. Two constants, two numbers. That's what you expect from a second order differential equation. The value of the function at the point, the derivative at a point. In fact, you are looking at a0 and a1 as the two constants that will determine a solution. And this is the value of h at 0. This is the derivative of h at 0.

So we can now write the following facts about the solution that we have found. So what do we know? That solutions fixed by giving a0 and a1. That correspond to the value of the function at 0 and the derivative of the function at 0.

And this gives one solution. Once you fix a0, you get a2, a4. And this is an even solution, because it has only even powers. And then from a1, you fixed a3, a5, all the other ones with an odd solution. OK.

Well, we solve the differential equation, which is really, in a sense, bad, because we were expecting that we can only solve it for some values of the energy. Moreover, you have a0, you get a2, a4, a6, a8. This will go on forever and not terminate. And then it will be an infinite polynomial that grows up and doesn't ever decline, which is sort of contradictory with the idea that we had before that near infinity the function was going to be some power, some fixed power, times this exponential. So this is what we're looking for, this h function now. It doesn't look like a fixed power. It looks like it goes forever.

So let's see what happens eventually when the coefficient, the value of the j index is large. For large j. aj plus 2 is roughly equal to, for large a, whatever the energy is, sufficiently large, the most important here is the 2j here, the j and the j. So you get 2 over j aj.

So roughly for large j, it behaves like that. And now you have to ask yourself the question, if you have a power series expansion whose coefficients behave like that, how badly is it at infinity? How about is it? You know it's the power series expansion because your h was all these coefficients. And suppose they behave like that. They grow in that way or decay in this way, because they're decaying. Is this a solution that's going to blow up? Or is it not going to blow up?

And here comes an important thing. This is pretty bad behavior, actually. It's pretty awful behavior. So let's see that. That's pretty bad. How do we see that? Well you could do it in different ways, depending on whether you want to derive that this is a bad behavior or guess it. I'm going to guess something.

I'm going to look at how does e to the u squared behave as a power series. Well, you know as a power series exponential is 1 over n u squared to the n. Here's n factorial. n equals 0 to infinity.

Now these two n's, u to the 2n, these are all even powers. So I'm going to change letters here, and I'm going to work with j from 0, 2, 4, over the evens. So I will write u to the j here. And that this correct, because you produce u to the 0, u to the 2, u to the fourth, these things. And j is really 2n, so here you will have one over j over 2 factorial. Now you might say, j over 2, isn't that a fraction? No, it's not a fraction, because j is even. So this is a nice factorial.

Now this is the coefficient, cj u to the j. And let's see how this coefficients vary. So this cj is 1 over j over 2 factorial. What is cj plus 2 over cj? Which is the analogue of this thing. Well, this would be 1 over j plus 2 over 2 factorial. And here is up there, so j over 2 factorial.

Well, this has one more factor in the denominator than the numerator. So this is roughly one over j over 2 plus 1, the last value of this. This integer is just one bigger than that. Now if j is large, this is roughly 1 over j over 2, which is 2 over j. Oh, exactly that stuff.

So it's pretty bad. If this series goes on forever, it will diverge like e to the u squared. And your h will be like e to the u squared with e to the minus u squared over 2 is going to be like e to the plus. u squared over 2 is going to go and behave this one. So it's going to do exactly the wrong thing.

If this series doesn't terminate, we have not succeeded. But happily, the series may terminate, because the j's are integers. So maybe for some energies that are integers, it terminates, and that's a solution. The only way to get a solution is if the series terminates. The only way it can terminate is that the e is some odd number over here. And that will solve the thing. So we actually need to do this. This shows the energy. You found why it's quantized.

So let's do it then. We're really done with this in a sense. This is the most important point of the lecture, is that the series must terminate, otherwise it will blow up horrendously. If it terminates as a polynomial, then everything is good.

So to terminate you can choose 2j plus 1 minus e to be 0. This will make aj plus 2 equal to 0. And your solution, your h of u, will begin. aj will be the last one that is non-zero, so it will be aj times u to the j, and it will go down like aj minus 2 u to the j minus 2. It will go down in steps of 2, because this recursion is always by steps of two.

So that's it. That's going to be the solution where these coefficients are going to be fixed by the recursive relation, and we have this. Now most people here call j equal n. So let's call it n. And then we have 2n plus 1 minus e equals 0. And h of u would be an u to the n plus all these things. That's the h.

The full solution is h times e to the minus u squared over 2 as we will see. But recall what e was. e here is 2n plus 1. But he was the true energy divided by h omega over two. That was long ago. It's gone. Long gone. So what have you found therefore? That the energy, that' we'll call en, the energy of the nth solution is going to be h omega over 2 2n plus 1. So it's actually h omega, and people write it n plus 1/2. Very famous result. The nth level of the harmonic oscillator has this energy.

And moreover, these objects, people choose these-- you see the constants are related by steps of two. So just like you could start with a0, or a1 and go up, you can go down. People call these functions Hermite functions. And they fix the notation so that this an is 2 to the n. They like it. It's a nice normalization.

So actually h of n is what we call the Hermite function of u sub n. And it goes like 2 to the n u to the n plus order u to the n minus 2 plus n minus 4, and it goes on and on like that.

OK, a couple things and we're done. Just for reference, the Hermite polynomial, if you're interested in it, is the one that solves this equation. And the Hermite sub n corresponds to e sub n, which is 2n plus 1. So the Hermite solution from that the equation is that the Hermite polynomial satisfies this minus 2u d Hn du plus 2n. Because en is 2n plus 1. So it's 2n Hn equals 0. That's the equation for the Hermite polynomial, and interesting thing to know.

Actually, if you want to generate the efficiently the Hermite polynomials, there's something called the generating function. e to the minus z squared plus 2zu. If you expand it in a power series of z, it actually gives you n equals 0 to infinity. If it's a power series of z, it will be some z to the n's. You can put a factor here n, and here is Hn of u. So you can use your mathematic program and expand this in powers of z. Collect the various powers of u that appear with z to the n, and that's Hn It's the most efficient way of generating Hn

And moreover, if you want to play in mathematics, you can show that such definition of Hn satisfies this equation. So it produces the solution.

So what have we found? Our end result is the following. Let me finish with that here. We had this potential, and the first energy level is called E0 and has energy h omega over 2. The next energy is E1. It has 3/2 h omega. Next one is E2 5/2 h omega. This polynomial is nth degree polynomial. So it has n zeros, therefore n nodes.

So these wave functions will have the right number of nodes. E0, the psi 0, will have no nodes. When you have psi 0, the Hn becomes a number for n equals zero. And the whole solution is the exponential of u squared over 2. The whole solution, in fact, is, as we wrote, psi n Hn of u e to the minus u squared over 2. In plain English, if you use an x, it will be Hn u with x over that constant a we had. And you have minus x squared over 2a squared. Those are your eigenfunctions. These are the solutions. Discrete spectrum, evenly spaced, the nicest spectrum possible. All the nodes are there. You will solve this in a more clever way next time.

[APPLAUSE]