PROFESSOR: All right, shall we get started? So, today-- well, before I get started-started-- so, let me open up to questions. Do y'all have questions from the last lecture, where we finished off angular momentum? Or really anything up to the last exam? Yeah?

AUDIENCE: So, what exactly happens with the half l states?

PROFESSOR: Ha, ha, ha! What happens with the half l states? OK, great question! So, we're gonna talk about that in some detail in a couple of weeks, but let me give you a quick preview. So, remember that when we studied the commutation relations, Lx, Ly is i h bar Lz . Without using the representation in terms of derivatives, with respect to a coordinate, without using the representations, in terms of translations and rotations along the sphere, right? When we just used the commutation relations, and nothing else, what we found was that the states corresponding to these guys, came in a tower, with either one state-- corresponding to little l equals 0-- or two states-- with l equals 1/2-- or three states-- with little l equals 1-- or four states-- with l equals 3/2-- and so on, and so forth. And we quickly deduced that it is impossible to represent the half integer states with a wave function which represents a probability distribution on a sphere.

We observed that that was impossible. And the reason is, if you did so, then when you take that wave function, if you rotate by 2pi-- in any direction-- if you rotate by 2pi the wave function comes back to minus itself. But the wave function has to be equal to itself at that same point. The value of the wave function at some point, is equal to the wave function at some point. That means the value of the wave function must be equal to minus itself. That means it must be zero0. So, you can't write a wave function-- which is a probability distribution on a sphere-- if the wave function has to be equal to minus itself at any given point. So, this is a strange thing.

And we sort of said, well, look, these are some other beasts.

But the question is, look, these furnish perfectly reasonable towers of states respecting these commutation relations. So, are they just wrong? Are they just meaningless? And what we're going to discover is the following-- and this is really gonna go back to the very first lecture, and so, we'll do this in more detail, but I'm going to quickly tell you-- imagine take a magnet, a little, tiny bar magnet. In fact, well, imagine you take a little bar magnet with some little magnetization, and you send it through a region that has a gradient for magnetic field. If there's a gradient-- so you know that a magnet wants to anti-align with the nearby magnet, north-south wants to go to south-north. So, you can't put a force on the magnet, but if you have a gradient of a magnetic field, then one end a dipole-- one end of your magnet-- can feel a stronger effective torque then the other guy. And you can get a net force.

So, you can get a net force. The important thing here, is that if you have a magnetic field which has a gradient, so that you've got some large B, here, and some smaller B, here, then you can get a force. And that force is going to be proportional to how big your magnet is. But it's also going to be proportional to the magnetic field. And if the force is proportional to the strength of your magnet, then how far-- if you send this magnet through a region, it'll get deflected in one direction or the other-- and how far it gets deflected is determined by how big of a magnet you sent through. You send in a bigger magnet, it deflects more. Everyone cool with that?

OK, here's a funny thing. So, that's fact one. Fact two, suppose I have a system which is a charged particle moving in a circular orbit. OK? A charged particle moving in a circular orbit. Or better yet, well, better yet, imaging you have a sphere-- this is a better model-- imagine you have a sphere of uniform charge distribution. OK? A little gelatinous sphere of uniform charge distribution, and you make it rotate, OK? So, that's charged, that's moving, forming a current. And that current generates a magnetic field along the axis of rotation, right? Right hand rule. So, if you have a charged sphere, and it's rotating, you get a magnetic moment. And how big is the magnetic moment, it's proportional to the rotation, to the angular momentum, OK?

So, you determine that, for a charged sphere here which is rotating with angular momentum, let's say, l, has a magnetic moment which is proportional to l. OK? So, let's put this together. Imagine we take a charged sphere, we send it rotating with same angular momentum, we send it through a field gradient, a gradient for magnetic field. What we'll see is we can measure that angular momentum by measuring the deflection. Because the bigger the angular momentum, the bigger the magnetic moment, but the bigger the magnetic moment, the bigger the deflection. Cool?

So, now here's the cool experiment. Take an electron. And electron has some charge. Is it a little, point-like thing? Is it a little sphere? Is it, you know-- Let's not ask that question just yet. It's an electron. The thing you get by ripping a negative charge off a hydrogen atom. So, take your electron and send it through a magnetic field gradient. Why would you do this? Because you want to measure the angular momentum of this electron. You want to see whether the electron is a little rotating thing or not. So, you send it through this magnetic field gradient, and if it gets deflected, you will have measured the magnetic moment. And if you have measured the magnetic moment, you'll have measured the angular momentum. OK? Here's the funny thing, if the electron weren't rotating, it would just go straight through, right? It would have no angular momentum, and it would have no magnetic moment, and thus it would not reflect. Yeah? If it's rotating, it's gonna deflect.

Here's the experiment we do. And here's the experimental results. The experimental results are every electron that gets sent through bends. And it either bends up a fixed amount, or it bends down a fixed amount. It never bends more, it never bends less, and it certainly never been zero. In fact, it always makes two spots on the screen. OK? Always makes two spots. It never hits the middle. No matter how you build this experiment, no matter how you rotate it, no matter what you do, it always hits one of two spots. What that tells you is, the angular momentum-- rather the magnetic moment-- can only take one of two values. But the angular momentum is just some geometric constant times the angular momentum. So, the angular momentum must take one of two possible values. Everyone cool with that?

So, from this experiment-- glorified as the Stern Gerlach Experiment-- from this experiment, we discover that the angular momentum, Lz, takes one of two values. L, along whatever direction we're measuring-- but let's say in the z direction-- Lz takes one of two values, plus some constant and, you know, plus h bar upon 2, or minus h bar upon 2. And you just do this measurement. But what this tells us is, which state? Which tower? Which set of states describe an electron in this apparatus? L equals 1/2.

But wait, we started off by talking about the rotation of a charged sphere, and deducing that the magnetic moment must be proportional to the angular momentum. And what we've just discovered is that this angular momentum-- the only sensible angular momentum, here-- is the two state tower, which can't be represented in terms of rotations on a sphere. Yeah? What we've learned from this experiment is that electrons carry a form of angular momentum, demonstrably. Which is one of these angular momentum 1/2 states, which never doesn't rotate, right? It always carries some angular momentum. However, it can't be expressed in terms of rotation of some spherical electron. It has nothing to do with rotations. If it did, we'd get this nonsensical thing of the wave function identically vanishes.

So, there's some other form of angular momentum-- a totally different form of angular momentum-- at least for electrons. Which, again, has the magnetic moment proportional to this angular momentum with some coefficient, which I'll call mu0. But I don't want to call it L, because L we usually use for rotational angular momentum. This is a different form of angular momentum, which is purely half integer, and we call that spin. And the spin satisfies exactly the same commutation relations-- it's a vector-- Sx with Sy is equal to ih bar Sz. So, it's like an angular momentum in every possible way, except it cannot be represented. Sz does not have any representation, in terms of h bar upon i [INAUDIBLE]. It is not related to a rotation. It's an intrinsic form of angular momentum. An electron just has it.

So, at this point, you ask me, look, what do you mean an electron just has it? And my answer to that question is, if you send an electron through a Stern Gerlach

Apparatus, it always hits one of two spots. And that's it, right? It's an experimental fact. And this is how we describe that experimental fact. And the legacy of these little L equals 1/2 states, is that they represent an internal form of angular momentum that only exists quantum mechanically, that you would have never noticed classically. That was a very long answer to what was initially a simple question. But we'll come back and do this in more detail, this was just a quick intro. Yeah?

AUDIENCE: So, for L equals 3/2, does that mean that there's 4 values of spins?

PROFESSOR: Yeah, that means there's [? 4 ?] values of spins. And so there are plenty of particles in the real world that have L equals 3/2. They're not fundamental particles, as far as we know. There are particles a nuclear physics that carry spin 3/2. There are all sorts of nuclei that carry spin 3/2, but we don't know of a fundamental particle. If super symmetry is true, then there must be a particle called a gravitino, which would be fundamental, and would have spin 3/2, and four states, but that hasn't been observed, yet. Other questions?

AUDIENCE: Was the [? latter of ?] seemingly nonsensical states discovered first, and then the experiment explain it, or was it the experiment--

PROFESSOR: Oh, no! Oh, that's a great question. We'll come back the that at end of today. So today, we're gonna do hydrogen, among other things. Although, I've taken so long talking about this, we might be a little slow. We'll talk about that a little more when we talk about hydrogen, but it was observed and deduced from experiment before it was understood that there was such a physical quantity. However, the observation that this commutation relation led to towers of states with this pre-existed as a mathematical statement. So, that was a mathematical observation from long previously, and it has a beautiful algebraic story, and all sort of nice things, but it hadn't been connected to the physics. And so, the observation that the electron must carry some intrinsic form of angular momentum with one of two values, neither of which is 0, was actually an experimental observation-- quasi-experimental observation-- long before it was understood exactly how to connect this stuff.

AUDIENCE: So it wasn't--?

PROFESSOR: I shouldn't say long, it was like within months, but whatever. Sorry.

AUDIENCE: The intent of the experiment wasn't to solve--

AUDIENCE: No, no. The experiment was this-- there are the spectrum-- Well, I'll tell you what the experiment was in a minute. OK, yeah?

AUDIENCE: [INAUDIBLE] Z has to be plus or minus 1/2. What fixes the direction in the Z direction?

PROFESSOR: Excellent. In this experiment, the thing that fixed the fact that I was probing Lz is that I made the magnetic field have a gradient in the Z direction. So, what I was sensitive to, since the force is actually proportionally to mu dot B-- or, really, mu dot the gradient of B, so, we'll do this in more detail later-- the direction of the gradient selects out which component of the angular momentum we're looking at. So, in this experiment, I'm measuring the angular momentum along this axis-- which for fun, I'll call Z, I could've called it X-- what I discover is the angular momentum along this axis must take one of two values.

But, the universe is rotationally invariant. So, it can't possibly matter whether I had done the experiment in this direction, or done the experiment in this direction, what that tells you is, in any direction if I measure the angular momentum of the electron along that direction, I will discover that it takes one of two values. This is also true of the L equals 1 states. Lz takes one of three values. What about Lx? Lx also takes one of three values, those three values. Is is every system in a state corresponding to one of those particular values? No, it could be in a superposition. But the eigenvalues, are these three eigenvalues, regardless of whether it's Lx, or Ly, or Lz. OK, it's a good thing to meditate upon.

Anything else? One more. Yeah?

AUDIENCE: [INAUDIBLE] the last problem [INAUDIBLE].

PROFESSOR: Indeed. Indeed. OK. Since some people haven't taken the-- there will be a conflict exam later today, so I'm not going to discuss the exam yet. But, very good observation, and not an accident.

OK, so, today we launch into 3D. We ditch our tricked-out tricycle, and we're gonna talk about real, physical systems in three dimensions. And as we'll discover, it's basically the same as in one dimension, we just have to write down more symbols. But the content is all the same. So, this will make obvious the reason we worked with 1D up until now, which is that there's not a heck of a lot more to be gained for the basic principles, but it's a lot more knowing to write down the expressions.

So, the first thing I wanted to do is write down the Laplacian in three dimensions in spherical coordinates-- And that is a beautiful abuse of notation-- in spherical coordinates. And I want to note a couple of things. So, first off, this Laplacian, this can be written in the following form, 1 over r dr r quantity squared. OK, that's going to be very useful for us-- trust me on this one-- this is also known as 1 over r dr squared r. And this, if you look back at your notes, this is nothing other than L squared-- except for the factor of h bar upon i-- but if it's squared, it's minus 1 upon h bar squared.

OK, so this horrible angular derivative, is nothing but L squared. OK, and you should remember the [? dd ?] thetas, and there are these funny sines and cosines. But just go back and compare your notes. So, this is an observation that the Laplacian in three dimensions and spherical coordinates takes this simple form. A simple radial derivative, which is two terms if you write it out linearly in this fashion, and one term if you write it this way, which is going to turn out to be useful for us. And the angular part can be written as 1 over r squared, times the angular momentum squared with a minus 1 over h bar squared. OK?

So, in just to check, remember that Lz is equal to h bar upon i d phi. So, Lz squared is going to be equal to minus h bar squared d phi squared. And you can see that that's one contribution to this beast. But, actually, let me-- I'm gonna commit a capital sin and erase what I just wrote, because I don't want it to distract you-- OK.

So, with that useful observation, I want to think about central potentials. I want to think about systems in 3D, which are spherically symmetric, because this is going to be a particularly simple class of systems, and it's also particularly physical. Simple things like a harmonic oscillator In three dimensions, which we solved in Cartesian coordinates earlier, we're gonna solve later, in spherical coordinates. Things like the isotropic harmonic oscillator, things like hydrogen, where the system is rotationally independent, the force of the potential only depends on the radial distance, all share a bunch of common properties, and I want to explore those. And along the way, we'll solve a toy model for hydrogen.

So, the energy for this is p squared upon 2m, plus a potential, which is a function only of the radial distance. But now, p squared is equal to minus h bar squared times the gradient squared. But this is gonna be equal to, from the first term, minus h bar squared-- let me just write this out-- times r dr squared r. And then from this term, plus minus h bar squared times minus 1 over h bar squared [? to L squared ?] [? over ?] r squared, plus L squared over r squared.

So, the energy can be written in a nice form. This is minus h bar squared, 1 upon r dr squared r-- whoops, sorry-- upon 2m, because it's p squared upon 2m. And from the second term, L squared over r squared upon 2m plus 1 over 2mr squared L squared plus u of r. OK, and this is the energy operator when the system is rotational invariant in spherical coordinates. Questions? Yeah?

AUDIENCE: [INAUDIBLE] is that an equals sign or minus?

PROFESSOR: This?

AUDIENCE: Yeah.

PROFESSOR: Oh, that's an equals sign. So, sorry. This is just quick algebra. So, it's useful to know it. So, consider the following thing, 1 over r dr r. Why would you ever care about such a thing? Well, let's square it. OK, because I did there. So, what is this equal to? Well, this is 1 over r dr r. 1 over r dr r. These guys cancel, right? 1 over r times dr. So, this is equal to 1 over r dr squared r. But, why is this equal to dr squared plus 2 over r times dr? And the answer is, they're operators. And so, you should ask how they act on functions. So, let's ask how they act on function. So, dr squared plus 2 over r dr times a function-- acting as a function-- is equal to f prime prime-- if this is a function of r-- plus 2 over r f prime. On the other hand, 1 over r dr squared r, acting on f of r, well, these derivatives can hit either the r of the f.

So, there's going to be a term where both derivatives hit f, in which case the rs cancel, and I get f prime prime. There's gonna be two terms where one of the d's hits this, one of the d's hits this, then there's the other term. So, there're two terms of that form. On d hits the r and gives me one, one d hits the f and gives me f prime. And then there's an overall 1 over r plus 2 over r f prime. And then there's a term were two d's hit the r, but if two d's hit the r, that's 0. So, that's it. So, these guys are equal to each other. So, why is this a particularly useful form? We'll see that in just a minute. So, I'm cheating a little bit by just writing this out and saying, this is going to be a useful form. But trust me, it's going to be a useful form. Yeah?

AUDIENCE: Do we need to find d squared [INAUDIBLE] dr squared r. Isn't that supposed to be 1 over r?

PROFESSOR: Oh shoot! Yes, that's supposed to be one of our-- Thank you. Thank you! Yes, over r. Thank you. Yes, thank you for that typo correction. Excellent. Thanks OK.

So, anytime we have a system which is rotationally invariant-- whose potential is rotationally invariant-- we can write the energy operator in this fashion. And now, you see something really lovely, which is that this only depends on r, this only depends on r, this depends on the angular coordinates, but only insofar as it depends on L squared. So, if we want to find the eigenfunctions of E, our life is going to be a lot easier if we work in eigenfunctions of L. Because that's gonna make this one [? Ex ?] on an eigenfunction of L, this is just going to become a constant.

So, now you have to answer the question, well, can we? Can we find functions which are eigenfunctions of E and of L, simultaneously? And so, the answer to that question is, well, compute the commutator. So, do these guys commute? In particular, of L squared. And, well, does L commute with the derivative with respect to r, L squared? Yeah, because L only depends on angular derivatives. It doesn't have any rs in it. And the rs don't care about the angular variables, so they commute. What about with this term? Well, L squared trivially commutes with itself and, again, r doesn't matter. And ditto, r and L squared commute. So, this is 0. These commute. So, we can find common eigenbasis. We can find a basis of functions which are eigenfunctions both of E and of L squared.

So, now we use separation. In particular, if we want to find a function-- an eigenfunction-- of the energy operator, E phi E is equal to E phi E, it's going to simplify our lives if we also let phi be an eigenfunction of the L squared. But we know what the eigenfunctions of L squared are. E phi E is equal to-- let me write this-- of r will then be equal to little phi of r times yLm of theta and phi.

Now, quickly, because these are the eigenfunctions of the L squared operator. Quick, is little l an integer or a half integer?

AUDIENCE: [MURMURS] Integer.

PROFESSOR: Why?

AUDIENCE: [MURMURS]

PROFESSOR: Yeah, because we're working with rotational angular momentum, right? And it only makes sense to talk about integer values of little l when we have gradients on a sphere-- when we're talking about rotations-- on a spherical coordinates, OK? So, little l has to be an integer.

And from this point forward in the class, any time I write l, I'll be talking about the rotational angular momentum corresponding to integer values. And when I'm talking about the half integer values, I'll write down s, OK? So, let's use this separation of variables. And what does that give us? Well, l squared acting on yLm gives us h bar squared lL plus 1. So, this tells us that E, acting on phi E, takes a particularly simple form. If phi E is proportional to a spherical harmonic, then this is gonna take the form minus h bar squared upon 2m 1 over r dr squared r plus 1 over 2mr squared l squared-- but l squared acting on the yLm gives us-- h bar squared lL plus 1, which is just a constant over r squared plus u of r phi E.

Question?

AUDIENCE: Yeah. [INAUDIBLE] yLm1 and yLm2?

PROFESSOR: Absolutely. So, can we consider superpositions of these guys? Absolutely, we can. However, we're using separation. So, we're gonna look at a single term, and then after constructing solutions with a single eigenfunction of L squared, we can then write down arbitrary superposition of them, and generate a complete basis of states. General statement about separation of variables. Other questions? OK.

So, here's the resulting energy eigenvalue equation. But notice that it's now, really nice. This is purely a function of r. We've removed all of the angular dependence by making this proportional to yLm. So, this has a little phi yLm, and this has a little phi yLm, and nothing depends on the little phi. Nothing depends on the yLm-- on the angular variables-- I can make this phi of r.

And if I want to make this the energy eigenvalue equation, instead of just the action of the energy operator, that is now my energy eigenvalue equation. This is the result of acting on phi with the energy operator, and this is the energy eigenvalue. Cool?

So, the upside here is that when we have a central potential, when the system is rotationally invariant, the potential energy is invariant under rotations, then the energy commutes with the angular momentum squared. And so, we can find common eigenfunctions. When we use separation of variable, the resulting energy eigenvalue equation becomes nothing but a 1D energy eigenvalue equation, right? This is just a 1D equation. Now, you might look at this and say, well, it's not quite a 1D equation, because if this were a 1D equation, we wouldn't have this funny 1 over r, and this funny r, right? It's not exactly what we would have got. It's got the minus h bar squareds upon 2m-- whoops, and there's, yeah, OK-- it's got this funny h bar squareds upon 2m, and it's got these 1 over-- or sorry,-- it's got the correct h bar squareds upon 2m, but it's got this funny r and 1 over r. So, let's get rid of that. Let's just quickly dispense with that funny set of r.

And this comes back to the sneaky trick I was referring to earlier, of writing this expression. So, rather than writing this out, it's convenient to write it in this form. Let's see why. So, if we have the E phi of r is equal to minus h bar squared upon 2m, 1 over r d squared r r, plus-- and now, what I'm gonna write is-- look, this is our potential, u of r. This is some silly, radial-dependent thing. I'm gonna write these two terms together, rather than writing them over, and over, and over again, I'm going to write them together, and call them V effective. Plus V effective of r, where V effective is just these guys, V effective.

Which has a contribution from the original potential, and from the angular momentum, which, notice the sign is plus 1 over r squared. So, the potential gets really large as you get to the origin. Phi of r. So, this r is annoying, and this 1 over r is annoying, but there's a nice way to get rid of it. Let phi of r-- well, this r, we want to get rid of-- so, let phi of r equals 1 over r u of r. OK, then 1 over r squared-- or sorry, 1 over r-- dr squared r phi is equal to 1 over r dr squared r times 1 over r times u, which is just u.

But meanwhile, V on phi is equal to-- well, V doesn't have any r derivatives, it's just a function-- so, V of phi is just 1 over r V on u. So, this equation becomes E on u, because this also picks up a 1 over r, is equal to minus h bar squared upon 2m dr squared plus V effective of r u of r. And this is exactly the energy eigenvalue equation for a 1D problem with the following potential. The potential, V effective of r, does the following two things-- whoops, don't want to draw it that way-- suppose we have a potential which is the Coulomb potential.

So, let's say, u is equal to minus E squared upon r. Just as an example. So, here's r, here is V effective. So, u first, so there's u-- u of r, so let me draw this-- V has another term, which is h bar squared lL plus 1 over 2mr squared. This is for any given value of l. This is a constant over r squared, with a plus sign. So, that's something that looks like this. This is falling off like 1 over r, this is falling off like 1 over r squared. So, it falls off more rapidly. And finally, can r be negative? No. It's defined from 0 to infinity. So, that's like having an infinite potential for negative r. So, our effective potential is the sum of these contributions-- wish I had colored chalk-- the sum of these contributions is going to look like this. So, that's my V effective. This is my Ll plus 1 [INAUDIBLE] squared over 2mr squared. And this is my u of r. Question?

AUDIENCE: [INAUDIBLE].

PROFESSOR: Good. OK, so this is u of r, the original potential.

AUDIENCE: OK.

PROFESSOR: OK? This is 1 over L squared-- or sorry-- lL 1 over 2mr squared.

AUDIENCE: [INAUDIBLE].

PROFESSOR: Oh shoot! Oh, I'm sorry! I'm terribly sorry! I've abused the notation terribly. Let's-- Oh! This is-- Crap! Sorry. This is standard notation. And in text, when I write this by hand, the potential is a big U, and the wave function is a little u. So, let this be a little u. OK, this is my little u and so, now I'm gonna have to-- oh jeez, this is horrible, sorry-- this is the potential, capital U with a bar underneath it. OK, seriously, so there's capital U with a bar underneath it. And here's V, which is gonna make my life easier, and this is the capital U with the bar underneath it. Capital U with the bar underneath it. Oh, I'm really sorry, I did not realize how confusing that would be. OK, is everyone happy with that? Yeah?

AUDIENCE: [INAUDIBLE].

PROFESSOR: Which one?

AUDIENCE: Middle. Middle.

PROFESSOR: Middle.

AUDIENCE: Up, up. Right there! Up! There.

PROFESSOR: Where?

AUDIENCE: To the right. [CHATTER] Near the eraser mark. [LAUGHTER]

PROFESSOR: So, these are the wave function.

AUDIENCE: I know.

PROFESSOR: That's the wave function. That is V.

AUDIENCE: [CHATTER]

PROFESSOR: Wait, if I erased, how can I correct it?

AUDIENCE: [CHATTER] There!

PROFESSOR: Excellent, so the thing that isn't here, would have a bar under it. Oh, oh, oh, oh, sorry! Ah! You wouldn't think it would be so hard. OK, good. And this is not [? related ?] to the wave function. OK, god, oh! That's horrible! Sorry guys, that notation is not obvious. My apologies. Oh, there's a better way to do this. OK, here's the better way to do this. Instead of calling the potential-- I'm sorry, your notes are getting destroyed now-- so instead of calling potential capital U, let's just call this V. Yeah.

AUDIENCE: [LAUGHTER] No!

PROFESSOR: And then we have V effective. No, no. This is good. This is good. We can be careful about this. So, this is V. This is V effective, which has V plus the angular momentum term. Oh, good Lord! This is V effective. This is V. V phi [INAUDIBLE] U. Good, this is V.

AUDIENCE: [INAUDIBLE] There's no U--

PROFESSOR: There's no U underline, it's now just V, V effective. Oh! Good Lord! OK, wow! That was an unnecessary confusion.

AUDIENCE: Top right.

PROFESSOR: Top right.

AUDIENCE: There is no bar.

[? PROFESSOR: Mu. ?]

AUDIENCE: Is that V or V effective?

PROFESSOR: That's V. Although, it would've been just as true as V effective. So, we can write V effective. It's true for both. Because it's just a function of r. Oh, for the love of God! OK. Let's check our sanity, and walk through the logic. So, the logic here is, we have some potential, it's a function only of r, yeah? As a consequence, since it doesn't care about the angles, we can write things in terms of the spherical harmonics, we can do separation of variables. Here's the energy eigenvalue equation. We discover that because we're working in spherical harmonics, the angular momentum term becomes just a function of r, with no other coefficients. So, now we have a function of r plus the potential V, this looks like an effective potential, V effective, which is the sum of these two terms. So, there's that equation.

On the other hand, this is tantalizingly close to but not quite the energy eigenvalue equation for a 1D problem with this potential, V effective. To make it obvious that it's, in fact, a 1D problem, we do a change of variables, phi goes to 1 over ru, and then 1 upon r d squared r phi becomes 1 over r d squared u, and V effective phi becomes 1 over r V effective u. Plugging that together, gives us this energy eigenvalue equation for u, the effective wave function, which is 1d problem. So, we can use all of our intuition and all of our machinery to solve this problem.

And now we have to ask, what exactly is the effective potential? And the effective potential has three contributions. First, it has the original V, secondly, it has the angular momentum term, which is a constant over r squared-- and here is that, constant over r squared-- and the sum of these is the effective. And this guy dominates because it's 1 over r squared. This dominates at small r, and this dominates at large r if it's 1 over r. So, we get an effective potential-- that I'll check-- there's the effective potential. And finally, the third fact is that r must be strictly positive, so as a 1D problem, that means it can't be negative, it's gotta have an infinite potential on the left.

So, as an example, let's go ahead and think more carefully about specifically this problem, about this Coulomb potential, and this 1D effective potential.

AUDIENCE: Professor?

PROFESSOR: Yeah?

AUDIENCE: [INAUDIBLE]?

PROFESSOR: Yes?

AUDIENCE: Where does the 1 over r go?

PROFESSOR: Good. So, remember the ddr squared term gave us a 1 over r out front. So, from this term, there should be 1 over r, here. From this term, there should also be a 1 over r. And from here, there should be a 1 over r.

AUDIENCE: Ah!

PROFESSOR: So, then I'm gonna cancel the 1 over r by multiplying the whole equation by r. Yeah? Sneaky, sneaky. So, any time you see-- any time, this is a general lesson-- anytime you see a differential equation that has this form-- two derivatives, plus 1 over r a derivative-- you know you can play some game like this. If you see this, declare in your mind a brief moment of triumph, because you know what technique to use. You can do this sort of rescaling by a power of r. And more generally, if you have a differential equation that looks like-- let me do this here-- if you have a differential equation that looks something like a derivative with respect to r plus a constant over r times phi, you know how to solve this. Let me say plus dot, dot, dot phi. You know how to solve this because ddr plus c over r means that phi, if there were nothing else, equals zero. If there were no other terms here, then this would say, ddr plus c over r is phi, that means when you take a derivative it's like dividing by r and multiplying by c. That means that phi goes like r to the minus c, right?

But if phi goes like r to the minus c, that's not the exact solution to the equation, but

I can write phi is equal to r to the minus c times u. And then this equation becomes ddr plus dot, dot, dot u equals zero. OK? Very useful little trick-- not really a trick, It's just observation-- and this is the second order version of the same thing. Very useful things to have in your back pocket for moments of need. OK?

So, let's pick up with this guy. So, let me give you a little name for this. So, this term that comes from the angular momentum [? bit, ?] this originally came from the kinetic energy, right? It came from the L squared over r, which was from the gradient squared energy. This is a kinetic energy term. Why is there a kinetic energy term? Well, what this is telling you is that if you have some angular momentum-- if little l is not equal to 0--- then as you get closer and closer to the origin, the potential energy is getting very, very large. And this should make sense. If you're spinning, and you pull in your arms, you have to do work, right? You have to pull those guys in. You speed up. You're increasing your kinetic energy due to conservation of angular momentum, right?

If you have rotationally invariance, as you bring in your hand you're increasing the kinetic energy. And so, this angular momentum barrier is just an expression of that. It's just saying that as you come to smaller and smaller radius, holding the angular momentum fixed, your velocity-- your angular velocity-- must increase-- your kinetic energy must increase-- and we're calling that a potential term just because we can. Because we've worked with definite angular momentum, OK? You should have done this in classical mechanics as well. Well, you should have done it in classical mechanics. So, this is called the angular momentum barrier.

Quick question, classically, if you take a charged particle around in a Coulomb potential, classically that system decays, right? Irradiates away energy. Does the angular momentum barrier save us from decaying? Is that why hydrogen is stable? No one wants to stake a claim here? Is hydrogen stable because of conversation of angular momentum?

AUDIENCE: No.

PROFESSOR: No. Absolutely not, right? So, first off, in your first problems set, when you did that calculation, that particle had angular momentum. So, and if can radiate that away through electromagnetic interactions. So, that didn't save us. Angular momentum won't save us. Another way to say this is that we can construct-- and we just explicitly see-- we can construct a state with which has little l equals 0. In which case the angular momentum barrier is 0 over r squared, because there's nothing. Angular momentum barrier's not what keeps you from decaying. And the reason is that the electron can radiate away energy and angular momentum, and so l will decrease and decrease, and can still fall down.

So, we still need a reason for why the hydrogen system, quantum mechanically, is stable. [? Why do ?] [? things exist? ?] So, let's answer that question. So, what I want to do now is, I want to solve-- do I really want to do it that way?-- well, actually, before we do, let's consider some last, general conditions. General facts for central potentials. So, let's look at some general facts for central potentials.

So, the first is, regardless of what the [? bare ?] potential was, just due to the angular momentum barrier, we have this 1 over r squared behavior near the origin. So, we can look at this, we can ask, look, what are the boundary conditions at the origin? What must be true of u of r near the origin? Near u of r-- or sorry, near r goes to zero-- what must be true of u of r? So, the right way to ask this question is not to look at this u of r, which is not actually the wave function, but to look at the actual wave function, phi sub E, which goes near r equals 0, like u of r over r.

So, what should be true of u? Can u diverge? Is that physical? Does u have to vanish? Can it take a constant value? So, I've given you a hint by telling you that I want to think about there being an infinite potential, but why? Why is that the right thing to do? Well, imagine u of r went to a constant value near the origin. If u of r goes to a constant value near the origin, then the wave function diverges near the origin. That's maybe not so bad, maybe it has a 1 over r singularity. It's not totally obvious that that's horrible. What's so bad about having a 1 over r behavior? So, suppose u goes to a constant. So, phi goes to constant over r. What's so bad about this?

So, let's look back at the kinetic energy. P is equal t-- the kinetic energy is gonna be minus h bar squared p squared-- so the energy is going to go like, p squared over 2md squared. But here's an important fact, d squared-- the Laplacian-- of 1 over r, well, it's easy to see what this is at a general point. At a general point, d squared has a term that looks like 1 over rd squared r r. So, 1 over rd squared r on 1 over r. Well, r times 1 over r, that's just 1. And this is 0, right? So, the gradient squared of 1 over r, is 0. Except, can that possibly be true at r equals 0? No, because what's the second derivative at 0? As you approach the origin from any direction, the function is going like 1 over r, OK, so it's growing, but it's growing in every direction. So, what's its first derivative at the origin? It's actually ill-defined, because it depends on the direction you come in.

The first direction coming in this way, the derivative looks like it's becoming this, from this direction it's becoming this, it's actually badly divergent. So, what's the second derivative? Well, the second derivative has to go as you go across this point, it's telling you how the first derivative changes. But it changes from plus infinity in this direction, to plus infinity in this direction. That's badly singular. So, this can't possibly be true, what I just wrote down here. And, in fact, d squared on 1 over r-- and this is a very good exercise for recitation-- is equal to delta of r. It's 0-- it's clearly 0 for r0 equals 0--- but at the origin, it's divergent. And it's divergent in exactly the way you need to get the delta function. OK, which is pretty awesome.

So, what that tells us is that if we have a wave function that goes like 1 over r, then the energy contribution-- energy acting on this wave function-- gives us a delta function at the origin. So, unless you have the potential, which is a delta function at the origin, nothing will cancel this off. You can't possibly satisfy the energy eigenvalue equation. So, u of r must go to 0 at r goes to 0. Because if it goes to a constant-- any constant-- we've got a bad divergence in the energy, yeah? In particular, if we calculate the energy, we'll discover that the energy is badly divergent. It does become divergent if we don't have u going to 0.

So, notice, by the way, as a side note, that since phi goes like, phi is equal to u over r, that means that phi goes to a constant. This is good, because what this is telling us is that the wave function-- So, truly, u is vanishing, but the probability density, which is the wave function squared, doesn't have to vanish. That's about the derivative of u, as you approach the origin from [? Lucatau's ?] Rule. So, this is the first general fact about central potential.

So, the next one-- and this is really fun one-- good Lord! Is that, note--- sorry, two more-- the energy depends on l but not on m. Just explicitly, in the energy eigenvalue equation, we have the angular momentum showing up int the effective potential, little l. But little m appears absolutely nowhere except in our choice of spherical harmonic. For any different m-- and this was pointing out before-- for any different m, we would've got the same equation. And that means that the energy eigenvalue can depend on l, but it can't depend on m, right? So, that means for each m in the allowed possible values, l, l minus 1, [? i ?] minus l-- and this is 2l plus 1 possible values-- for each of these m's, the energy is the same.

And I'll call this E sub l, because the energy can depend on l. Why? The degeneracy of E sub L is equal to 2l plus 1. Why? Why do we have this degeneracy?

AUDIENCE: [INAUDIBLE].

PROFESSOR: Yeah, exactly. We get the degeneracies when we have symmetries, right? When we have a symmetry, we get a degeneracy. And so, here we have a degeneracy. And this degeneracy isn't fixed by rotational invariance. And why is this the right thing? Rotational symmetry? So why, did this give us this degeneracy? But what the rotational degeneracy is saying is, look, if you've got some total angular momentum, the energy can't possibly depend on whether most of it's in Z, or most of it's in X, or most of it's and Y. It can't possibly depend on that, but that's what m is telling you. M is just telling you what fraction is contained in a particular direction.

So, rotational symmetry immediately tells you this. But there's a nice way to phrase this, which of the following, look, what is rotational symmetry? Rotational symmetry is the statement that the energy doesn't care about rotations. And in particular, it must commute with Lx, and with Ly, and with Lz. So, this is rotationally symmetry. And I'm going to interpret these in a nice way.

So, this guy tells me I can find common eigenfunctions. And, more to the point, a full common eigenbasis of E and Lz. Can I also find a common eigenbasis of ELz and Ly? Are there common eigenvectors of ELz and Ly?

AUDIENCE: [CHATTER]

PROFESSOR: No. Are there common eigenfunctions of Lz and Ly?

AUDIENCE: No.

PROFESSOR: No, because they don't commute, right? E commutes with each of these. OK, so, I'm just going to say, I'm gonna pick a common eigenbasis of E and Lz-- but I could've picked Lx, or I could've picked Ly, I'm just picking Lz because that's our convention-- but what do these two-- Once I've chosen this-- I'm gonna work with a common eigenbasis of E and Lz-- what do these two commutators tell me? These two commutators tell me that E commutes with Lx plus iLy and Lx minus iLy, L plus/minus. So, this tells you that if you have an eigenfunctions of E, and you act with a raising operator, you get another eigenfunction of E. And thus, we get our 2L plus 1 degeneracy, because we can walk up and down the tower using L plus and L minus. Cool? OK.

So, this is a nice example that when you have a symmetry you get a degeneracy, and vice versa.

OK. So, let's do some examples of using these central potentials.

AUDIENCE: Professor?

PROFESSOR: Yeah

AUDIENCE: [INAUDIBLE]?

PROFESSOR: It's 0. So, E with Lx is 0. E with Ly-- So, are you happy with that statement? That E with Lx is 0?

AUDIENCE: Yeah.

PROFESSOR: Yeah. Good. OK, and so this 0 because this is just Lx plus iOi. So, E with Lx is 0, and E with Ly is 0, so E commutes with these guys. And so, this is like the statement that L squared with L plus/minus equals 0.

AUDIENCE: [INAUDIBLE].

PROFESSOR: Cool. OK, so, let's do some examples. So, the first example is gonna be-- actually, I'm going to skip this spherical well example-- because it's just not that interesting, but it's in the notes, and you really need to look at it. Oh hell, yes, I'm going to do it. OK, so, the spherical well. So, I'm going to do it in an abridged form, and maybe it's a good thing for recitation.

AUDIENCE: Professor?

PROFESSOR: Thank you recitation leader. So, in this spherical well, what's the potential? So, here's v of r. Not U bar, and not V effective, just v or r. And the potential is going to be this, so, here's r equals 0. And if it's a spherical infinite well, then I'm gonna say, the potential is infinite outside of some distance, l. OK? And it's 0 inside. So, what does this give us? Well, in order to solve the system, we know that the first thing we do is we separate out with yLms, and then we re-scale by 1 over r to get the function of u, and we get this equation, which is E on u is equal to minus h bar squared upon 2m dr squared and plus v effective-- well, plus [? lL ?] plus 1-- over r squared with a 2m and an h bar squared. And the potential is 0, inside. So we can just write this.

So, if you just-- let me pull out the h bar squareds over 2m-- it becomes minus [INAUDIBLE] plus 1 over r squared. So, this is not a terrible differential equation. And one can do some good work to solve it, but it's a harder differential equation than I want to spend the time to study right now, so I'm just going to consider the case-- special case-- when there's zero angular momentum, little l equals 0. So, in the special case of a l equals 0, E-- and I should call this u sub l-- Eu sub 0 is equal to h bar squared upon 2m. And now, this term is gone-- the angular momentum barrier is gone-- because there's no angular momentum, dr squared ul. Which can be written succinctly as ul-- or sorry, u0-- prime prime, because this is only a function of r.

So now, this is a ridiculously easy equation. We know how to solve this equation, right? This is saying that the energy, a constant, times u is two derivatives times this constant. So, u0 can be written as a cosine of kx-- or sorry-- kr plus b sine of kr, where h bar squared k squared upon 2m equals E. And I should really call this E sub 0, because it could depend on little l, here. So, there's our momentary solution, however, we have to satisfy our boundary conditions, which is that it's gotta vanish at the origin, but it's also gotta vanish at the wall.

So, the boundary conditions, u of 0 equals 0 tells us that a must be equal to 0, and u of l equals 0 tells us that, well, if this is 0, we've just got B, but sine of kr evaluated at l, which is sine of kl, must be equal to 0. So, kl must be the 0 of sine, must be n pi over-- must be equal to n pi, a multiple of pi. And so, this tells you what the energy is. So, this is just like the 1D system. It's just exactly like when the 1D system.

So now, to finally close this off. What does this tell you that the eigenfunctions are? And let me do that here. So, therefore, the wave function phi sub E0 of r theta and phi-- oh god, oh jesus, this is so much easier in [INAUDIBLE] so, phi [INAUDIBLE] 0 of r theta and phi is equal to y0m. But what must m be? 0, because m goes from plus L to minus L, 0. I'm just [INAUDIBLE] the argument. Y00 times, not u of r, times 1 over r times u. 1 over r times u of r. But u of r is a constant times sine of kr. Sine of kr, but k is equal to n pi over L. N pi over Lr. And what's Y00? It's a constant. And so, there's an overall normalization constant, that I'll call n.

OK, so, we get that our wave function is 1 over r times sine of n pi over Lr. So, this looks bad. There's a 1 over r. Why is this not bad? At the origin, why is this not something I should worry about it?

AUDIENCE: [MURMURS]

PROFESSOR: Yeah, because sine is linear, first of all, [INAUDIBLE] argument. So, this goes like, n pi over L times r. That r cancels the 1 over r. So, near the origin, this goes like a constant. Yeah? So, u has to 0, but the wave function doesn't. Cool? OK. So, this is a very nice more general story for larger L, which I hope you see in the recitation. OK. Questions on the spherical well? The whole point here-- Oh, yeah, go.

AUDIENCE: What do [INAUDIBLE] generally [INAUDIBLE]?

PROFESSOR: That's true. So, good. So, let me rephrase the question, and tell me if this is the same question. So, this is strange. There's nothing special about the origin. So, why do I have a 0 at the origin? Is that the question?

AUDIENCE: Yeah.

PROFESSOR: OK. It's true. There's nothing special about the origin, except for two things. One thing that's special about the origin is we're working in a system which has a rotational symmetry. But rotational symmetry is rotational symmetry around some particular point. So, there's always a special central point anytime you have a rotational symmetry. It's the point fixed by the rotations. So, actually, the origin is a special point here. Second, saying that little u has a 0 is not the same as saying that the wave function has a 0. Little u has a 0, but it gets multiplied by 1 over r. So, the wave function, in fact, is non-zero, there. So, the physical thing is the probability distribution, which is the [? norm ?] squared of the wave function. And it doesn't have a 0 at the origin. Does that satisfy?

AUDIENCE: Yes.

PROFESSOR: OK. So, the origin is special when you have a central potential. That's where the proton is, right? Right, OK. So, there is something special about the origin. Wow, that was a really [? anti-caplarian ?] sort of argument. OK, so that's where the proton-- so, there is something special about the origin, and the wave function doesn't vanish there, even if u does. It may vanish there, but it doesn't necessarily have to. And we'll see that in a minute. Other questions? Yeah?

AUDIENCE: So, what again, what's the reasoning for saying that the u of r has to vanish at [? 0 instead ?] [? of L? ?]

PROFESSOR: Good. The reason that u of r had to vanish at the origin is that if it doesn't vanish at the origin, then the wave function diverges-- whoops, phi goes to constant-- if u doesn't go to 0, if it goes to any constant, non-zero, then the wave function diverges. And if we calculate the energy, we get a delta function at the origin. So, there's an infinite contribution of energy at the origin. That's not physical. So, in order to get a sensible wave function with finite energies, we need to have the u vanishes, because of the 1 over u. And the reason that we said it had to vanish at l, was because I was considering this spherical well-- spherical infinite well-- where a particle is stuck inside a region of radius, capital L, and that's just what I mean by saying I have an infinite potential.

AUDIENCE: OK, thanks.

PROFESSOR: Cool? Yeah. Others? OK.

So, with all that done, we can now do the hydrogen-- or the Coulomb-- potential. And I want to emphasize that we often use the following words when-- people often use the following words when solving this problem-- we will now solve the problem of hydrogen. This is false. I am not about to solve for you the problem of hydrogen. I am going to construct for you a nice toy model, which turns out to be an excellent first pass at explaining the properties observed in hydrogen gases, their emission spectra, and their physics. This is a model. It is a bad model. It doesn't fit the data. But it's pretty good. And we'll be able to improve it later. OK?

So, it is the solution of the Coulomb potential. And what I want to emphasize to you, I cannot say this strongly enough, physics is a process of building models that do a good job of predicting. And the better their predictions, the better the model. But they're all wrong. Every single model you ever get from physics is wrong. There are just some that are less stupidly wrong. Some are a better approximation to the data, OK? This is not hydrogen. This is going to be our first pass at hydrogen. It's the Coulomb potential. And the Coulomb potential, V of r, is equal to minus e squared over r.

This is what you would get if you had a classical particle with infinite mass and charge plus b. And then another particle over here, with mass, little m, and charge, minus e. And you didn't pay too much attention to things like relativity, or spin, or, you know, lots of other things. And you have no background magnetic field, or electric field, and anything else. And if these are point particles, and-- All of those things are false that I just said. But if all those things were true, in that imaginary universe, this would be the salient problem to solve. So, let's solve it. Now, are all those things that I said that were false-- the proton's a point particle, the proton is infinitely massive, there's no spin-- are those preposterously stupid?

AUDIENCE: No.

PROFESSOR: No, they're excellent approximations in a lot of situations. So, they're not crazy wrong. They're just not exactly correct. I want to keep this in your mind. These are gonna be good models, but they're not exact. So, we're not solving hydrogen, we're gonna solve this idealized Coulomb potential problem.

OK, so let's solve it. So, if V is minus e over r squared, then the equation for the rescaled wave function, u, becomes minus h bar squared upon 2mu prime prime of r plus the effective potential, which is h bar squared upon 2mll plus 1 over r squared minus e squared over r u is equal to e sub l u. So, there's the equation we want to solve. We've already used separation of variables, and we know that the wave function is this little u times 1 over r times yLm, for some l and some m. So, the first thing we should do any time you're solving an interesting problem, the first thing you should do is do dimensional analysis. And if you do dimensional analysis, the units of e squared-- well, this is easy-- e squared must be an energy times a length. So, this is an energy times a length. Also known as p squared l, momentum squared over 2m, 2 times the mass times the length. It's useful to put things in terms of mass, momentum, and lengths, because you can cancel them out. H bar has units of p times l. And what's the only other parameter we have? We have the mass, which has units of mass. OK.

And so, from this, we can build two nice quantities. The first, is we can build r0. We can build something with units of a radius. And I'm going to choose the factors of 2 judiciously, h bar squared over 2me squared-- whoops, e squared-- so, let's just make sure this has the right units. E squared has units of energy times the length, but h bar squared over 2m has units of p squared l squared over 2m, so that has units of energy times the length squared. So, length squared over length, this has units of length, so this is good. So, there's a parameter that has units of length.

And from this, it's easy to see that we can build a characteristic energy by taking e squared and dividing it by this length scale. And so then, the energy, which I'll call e0, which is equal to e squared over r0, is equal to 2me to the 4th over h bar squared. So, before we do anything else, without solving any problems, we immediately can do a couple of things. The first is, if you take the system and I ask you, look, what do you expect? If this is a quantum mechanical-- a 1d problem in quantum mechanics-- with a potential, and we know something about 1D quantum mechanical problems-- I guess, this guy-- we know something about 1D quantum mechanical problems. Which is that the ground state has what energy? Some finite energy. It doesn't have infinite negative energy. It's got some finite energy. What do you expect to be roughly the ground state energy of this system?

AUDIENCE: [MURMURING]

PROFESSOR: Yeah. Right. Roughly minus e0. That seems like a pretty good guess. It's the only dimensional sensible thing. Maybe we're off by factors of 2. But, maybe it's minus e0. So, that's a good guess, a first thing, before we do any calculation. And if you actually take mu e to the 4th over h bar squared, this is off by, unfortunately, a factor of 4. This is equal to 4 times the binding energy, which is also called the Rydberg constant. Wanna make sure I get my factors of two right. Yep, I'm off by a factor of 4. I'm off by a factor of 4 from what we'll call the Rydberg energy, which is 13.6 eV. And this is observed binding energy of hydrogen.

So, before we do anything, before we solve any equation, we have a fabulous estimate of the binding energy of hydrogen, right? All the work we're about to do is gonna be to deal with this factor of 4, right? Which, I mean, is important, but I just want to emphasize how much you get just from doing dimensional analysis.

Immediately upon knowing the rules of quantum mechanics, knowing that this is the equation you should solve, without ever touching that equation, just dimensional analysis gives you this answer. OK? Which is fabulous.

So, with that motivation, let's solve this problem. Oh, by the way, what do you think r0 is a good approximation to? Well, it's a length scale.

AUDIENCE: [INAUDIBLE].

PROFESSOR: Yeah! It's probably something like the expectation value of the radius-- or maybe of the radius squared-- because the expectation value of the radius is probably 0. OK, so, let's solve this system. And at this point, I'm not gonna actually solve out the differential equation in detail. I'm just gonna tell you how the solution goes, because solving it is a sort of involved undertaking. And so, here's the first thing, so we look at this equation. So, we had this differential equation-- this guy-- and we want to solve it. So, think back to the harmonic oscillator when we did the brute force method of solving the hydrogen system, OK? When we did the brute force method-- she sells seashells-- when the brute force method of solving, what did we do? We first did, we did asymptotic analysis. We extracted the overall asymptotic form, at infinity and at the origin, to get a nice regular differential equation that didn't have any funny singularities, and then we did a series approximation. OK?

Now, do most differential equations have a simple closed form expression? A solution? No, most differential equations of some, maybe if you're lucky, it's a special function that people have studied in detail, but most don't have a simple solution like a Gaussian or a power large, or something. Most of them just have some complicated solution. This is one of those miraculous differential equations where we can actually exactly write down the solution by doing the series approximation, having done asymptotic analysis. So, the first thing when doing dimensional analysis too, let's make everything dimensionless.

OK, and it's easy to see what the right thing to do is. Take r and make it dimensionless by pulling out a factor of rho, or of r0. So, I'll pick our new variable is gonna be called rho, this is dimensionless. And the second thing is I want to take the energy, and I will write it as minus e0, times some dimensionless energy, epsilon. So, these guys are my dimensionless variables. And when you go through and do that, the equation you get is minus d rho squared plus l l plus 1 over rho squared minus 1 over rho plus epsilon u is equal to 0. So, the form of this differential equation is, OK, it's not different in any deep way, but it's a little bit easier. This is gonna be the easier way to deal with this, because I don't have to deal with any stupid constant.

And so now, let's do the brute force thing. Three, asymptotic analysis. And here, I'm just going to write down the answers. And the reason is, first off, this is something you should either do in recitation, or see-- go through-- on your own, but this is just the mathematics of solving a differential equation. This is not the important part. So, when rho goes to infinity, which terms dominate? Well, this is not terribly important. This is not terribly important. That term is gonna dominate. And if we get that d rho squared plus u, rho goes to infinity, these two terms dominate. Well, two derivatives is a constant. You know what those solutions look like, they look like exponentials, with the exponential being brute-- with the power-- the exponent, sorry, being root epsilon. So, u is going to go like e to the minus square root epsilon rho.

For normalize-ability, I picked the minus, I could've picked the plus, that would've been divergent. So, as rho goes to 0, what happens? Well, as rho goes to 0, this is insignificant. And this totally dominates over this guy. On the other hand, if l is equal to 0, then this is the only term that survives, so we'd better make sure that that behaves gracefully. As rho goes to 0, asymptotic analysis is gonna tell us that u goes like rho. Well, two derivatives, we pulled down a rho squared, and so two derivatives in this guy, we pulled down an l, then an l plus 1. So, this should go like rho to the l plus 1.

There's also another term. So, in the same way that there were two solutions to this guy asymptotically-- one growing, one decreasing-- here, there's another solution, which is rho to the minus l. That also does it, because we get minus l, then minus minus l minus 1, which gives us the plus l l plus 1. But that is also badly diversion at the origin, it goes like 1 over 0 to the l. That's bad. So, these are my solutions.

So, this tells us, having done this in analysis, we should write that u is equal to rho to the l plus 1 times e to the minus root epsilon rho times some remaining function, which I'll call v, little v. Little v of rho, and this, asymptotically, should go to a constant near the origin and something that vanishes slower than an exponential at infinity.

So then, we take this and we do our series expansion. So, we take that expression, we plug it in. At that point, all we're doing is a change of variables. We plug it in, and we get a resulting differential equation. Rho v prime prime plus 2 1 plus l minus root epsilon rho v prime plus 1 minus 2 root epsilon l plus 1 v equals 0. So, this is the resulting differential equation for the little v guy. And we do a series expansion. V is equal to sum over, sum from j equals 0 to infinity, of a sub j rho to the j. Plug this guy in here, just like in the case of the harmonic oscillator equation, and get a series expansion.

Now, OK, let me write it out this way. And the series expansion has a solution, which is a sub j plus 1. And this is, actually, kind of a fun process. So, if you, you know, like quick little calculations, this is a sweet little calculation to take this expression. Plug it in and derive this recursion relation, which is root-- or 2 root-- epsilon times j plus l plus 1 minus 1 over j plus 1 j plus 2 l plus 2 aj. So, here's our series expansion. And in order for this terminate, we must have that some aj max plus 1 is equal to 0. So, one of these guys must eventually vanish. And the only thing's that's changing is little j. So, what that tells us is that for some maximum value of little j, root 2 epsilon times j maximum plus l plus 1 is equal to minus 1.

But that gives us a relationship between overall j max, little l, and the energy. And if you go through, what you discover is that the energy is equal to 1 over 4 n squared, where n is equal to j max plus l plus 1. And what this tells is that the energy is labeled by an integer, n, and an integer, l, and an integer, m-- these are from the spherical harmonics, and n came from the series expansion-- and it's equal to minus e0 over 4 n squared, independent of l and m.

And so, by solving the differential equation exactly, which in this case we kind of amazingly can, what we discover is that the energy eigenvalues are, indeed, exactly 1/4 of e0. And they're spaced with a 1 over n squared, which does two things. Not only does that explain-- so, let's think about the consequence of this very briefly-- not only does that explain the minus 13.6 eV, not only does that explain the binding energy of hydrogen as is observed, that it does more. Remember in the very beginning one of the experimental facts we wanted to explain about the universe was that the spectrum of light of hydrogen went like 30 over 4 n squared. This was the Rydberg relation. And now we see explicitly. So, we've solved for that expansion.

But there's a real puzzle here. Purely on very general grounds, we derived earlier that when you have a rotationally invariant potential-- a central potential-- every energy should be degenerate, with degeneracy 2l plus 1. It can depend on l, but it must be independent of m. But here, we've discovered-- first off, we've fit a nice bit of experimental data, but we've discovered the energy is, in fact, not just independent of m, but it's independent of l, too. Why? What symmetry is explaining this extra degeneracy?

We'll pick that up next time.