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PROFESSOR: Hi everyone. I'm in a very good mood today. It's nothing to do with the class, but I'm having a baby.

[APPLAUSE]

PROFESSOR: So that's kind of exciting. So if I just started giggling, you'll know why. And in six months if I am just weeping and on the ground, you'll also know why.

So today we're going to do two things. The first is I'm going to give you-- well, the first is review a little bit of practice for the exam we're going to have on Thursday. So let me tell you a little bit more about the exam. The exam, by the way, has been rescheduled to be in the 6120, not in Walker Gym. So it's going to be the usual place, we're not moving.

And the reason is I'm changing the format on this exam, in part to make it a little less of a burden to everyone. But also in part because I've been struggling with the question of how to make the exam most useful. The purpose of an exam like this is not to get grades for you guys, although that's an incidental byproduct.

The purpose is to give you some feedback on how you're doing, how your command of the material has evolved. And also to help you learn some of the things that you might not have mastered. So the way the exam is going to be structured is going about 15 minutes of short answer questions-- a couple of very short computations but mostly short answer questions-- on paper. You'll hand those back, and then we'll go over those questions in class afterwards.

So it's a relatively low pressure exam and it's mostly conceptual. It will cover everything we've done through this last problem set to the degree that we get to the

lecture part of today after questions. Today's lecture will not be directly covered, however it will be fair game for the next midterm-- which will be more of a traditional midterm-- and that's coming in April.

So the structure of today is, I'm going to give you a whole bunch of clicker questions. So make sure you've got your clickers out. And those clicker questions are going to give you a sense for the level and scope of the exam.

The exam will be a little harder than the clicker questions, but not a whole lot. And the difference is just that it's going to be on paper in front of you instead of clickered. And thus, that gives you a little more room to do calculations on a piece of paper, short calculations.

And then after that we'll come on to-- in some sense a review, but also an introduction to the Dirac Bra-Ket notation that many of our textbooks use, but that we haven't introduced in lectures so far. Any questions before we get started?

AUDIENCE: What channel are we?

PROFESSOR: 41. Other questions?

AUDIENCE: So no practice exam?

PROFESSOR: I think there will probably not be a practice exam because of the shift in format. Today will basically be your practice exam. If I can get my work together and get you guys a practice exam of the right format, then maybe. But it wouldn't be all that useful is the problem. So watch what happens today.

AUDIENCE: Will you post the clicker questions from today and the last time on the website?

PROFESSOR: I will post the questions from today on the website, yes.

AUDIENCE: So it's 50 minutes, short answer. Is there a certain number of questions that's going to be on it?

PROFESSOR: I'm not going to tell you that. But it's not going to be a time trial. You're not going to

be racing to get the, you know.

AUDIENCE: Will it still be worth as much as an exam?

PROFESSOR: It will. Because to my mind, part of the reason to design the exam this way is that it's testing your conceptual understanding, which is the important thing. So it will be. Other questions?

All right. So let's get started with the clicker part of today. So we're on channel 41. So consider this eigenvalue equation, two derivatives on f -- it's constant-- times f of x . How many of these are eigenfunctions with the corresponding eigenvalue?

You've got about 10 seconds. So go ahead and put in your clicks. And so your-- whoops, oh sorry. That just cleared your responses, unfortunately. Don't worry, they're saved in memory. It just cleared them off my screen.

So the responses-- here we go. Oops, that didn't work. Wow, it just totally disappeared out of that app. Wow, that's so weird.

Oh no, that didn't work. Let's try this. Where did it go? Ah, there it is. Whoa, this is all very confusing.

So that was your response, B and C, 63% and 38%. So let's go back to this. Quickly discuss this with your neighbor. And now go ahead and click in your new answers if you-- you can keep talking, that's fine.

Another five seconds. Awesome. OK, that's it. So the answer is B. And 99% of you all got that.

I suspect someone didn't click. OK good, next question. ΨI and ΨII are two solutions of the Schrodinger equation.

Is the sum of the two of them with coefficients A and B also a solution of the Schrodinger equation? Oh, and I forgot to start the clicky thingy. So click now.

AUDIENCE: Did you mean to say B $\Psi 2$ on that thing?

PROFESSOR: Oh yeah, it is supposed to say $B \Psi_2$. It says $B \Psi_2$ on the next-- sorry, it's $A \Psi_1$ plus $B \Psi_2$. It should say $A \Psi_1$ plus $B \Psi_2$. Thank you.

AUDIENCE: Does it mean that the same Schrodinger equation with the same potential?

PROFESSOR: Yeah, with the same potential. Yeah, Yeah.

[LAUGHTER]

PROFESSOR: Wow, Yeah. So you know Einstein said, God didn't play dice. And let me paraphrase that as, God doesn't mess with you in clicker questions. And you guys have effectively universally that the answer is A, yes, superposition principle.

OK next question is going to be for answer-- here we go. Consider an infinite square well with width a . And compared to the infinite square well with A , the ground state of a finite well is lower, higher, same energy, or undetermined.

You've got 10 seconds, so continue thinking through this. 5 seconds. OK, and you mostly got it, but have a quick chat with the person next to you.

All right, let's try again. Such a good technique. All right, another five seconds to put in your answer. 4-3-2-1. And you virtually all got it right, a lower energy.

And let's just think about this intuitively. Intuitively, the gradient of the potential is the force, right? So in the second case, you've got less force cramming the article inside the box so it's being squeezed less tightly.

More physically, you see that there is an evanescent tail on the outside. What that tells you is the wave function didn't have to go to zero at the ends. It just had to get small and latch onto a dying exponential. That's from the qualitative analysis of wave functions.

But meanwhile, what that tells you is, it has to curve less inside in order-- it doesn't have to get to zero, it just has to get to a small value where it matches to the decaying exponential. So if the curvature is less, then the energy is less. Cool?

OK, next question. So any questions on that before I? Good.

Time zero wave function infinite well with a is this, sine squared with a normalization. What's the wave function at a subsequent time t ? I will remind you that you have solved the problem of the infinite square well, and you know what the eigenenergies and eigenfunctions are of the energy operator in the infinite square well. So remember back to what those are.

All right, and you have five seconds. OK, we are at about 50-50 correct. So chat with the person next to you. All right. And now, any moment, go ahead and vote again.

Good, five more seconds and then put in your final vote. OK, that's it for now. So what's the answer? D is the answer, but a lot of people still had some doubts. So who wants to give an explanation for why it's D?

AUDIENCE: So sine squared is not an eigenfunction.

PROFESSOR: Fantastic.

AUDIENCE: So in some way it has to be a summation of eigenfunctions. So not even having to know what the eigenfunctions are, there's only one summation in it.

PROFESSOR: Excellent, excellent.

AUDIENCE: And if you do know what the eigenfunctions are, now you know that [INAUDIBLE].

PROFESSOR: Brilliant, so I'm going to restate that. That was exactly correct in every step. So the first thing is, that wave function at time 0, sine squared of x , is not an eigenfunction of the energy operator for the system. In fact, we've computed the eigenfunctions for the energy operator in the infinite square well, and they're sines where they get a zero at the ends.

On the other hand, any function satisfying the boundary conditions-- normalizable, hits zero at the boundaries-- is a superposition of energy eigenfunctions, and we can use that to determine the time evolution we take that superposition and add on

a phase-- $e^{-iEt/\hbar}$ for each of the energy eigenfunctions. And in part D we express that wave function as a superposition with the coefficient c_n determined from the overlap of our original function and the energy eigenfunctions.

Everyone cool with that? So this is literally just a transcription of one of our postulates. OK, questions? If you have any questions at all, ask them. This is the time to ask them. Yeah.

AUDIENCE: So the way get c_n [INAUDIBLE]?

PROFESSOR: Exactly. So the way you get c_n is by saying, look I have my wave function, $\psi(x)$ is equal to sum over n $c_n \phi_n(x)$, where these ϕ_n 's energy eigenfunctions. E_{ϕ_n} is equal to E_n . And we also know that the integral of ϕ_n 's complex conjugate ϕ_m is equal to δ_{mn} . This is the statement that they're orthogonal and properly normalized.

We also write this as $\psi(x) = \sum_n c_n \phi_n(x)$. And we can use this to determine the c_n is equal to the inner product of ϕ_n with our wave function. This is equal to the integral $\int dx \phi_n^*(x) \psi(x)$, which is equal to the integral $\int dx \phi_n^*(x) \sum_m c_m \phi_m(x)$.

But this sum over m of c_m can be pulled out because this is just an integral over a sum of terms, which is the same as the sum over m $c_m \int dx \phi_n^*(x) \phi_m(x)$. And that's δ_{nm} , which is 0 unless n is equal to m , because these guys are orthogonal and properly normalized. So this is zero unless n is equal to m . So in the sum, the only term that contributes is when m is equal to n , this is equal to c_n .

So c_n is given by the overlap of our wave function with the corresponding eigenfunction. This allows us to take our function, a known function-- for example, $\sin^2(x)$ -- and express the coefficients in terms of an overlap of our wave function $\sin^2(x)$ with the wave functions $\sin(x)$. And that's exactly the expression you see below, c_n is equal to the integral of $\sin^2(x)$ times $\sin(x)$, which is the energy eigenfunction.

Other questions? OK, great. Next problem. Come to me computer.

Why can they just be written in Python. OK, good. the eigenstates ϕ_n -- which we usually call ψ_n , but there it is-- form an orthonormal set. Meaning integral of $\phi_m^* \phi_n$ is δ_{mn} . What is the value of integral ψ_m against the sum $\sum c_n \phi_n$?

You've got five seconds. I'll give you a little extra time, because people are clicking away. OK you now have seven seconds, because time is non-linear. OK, so quickly discuss, because there's still some ambiguity here.

All right, you have 10 seconds to modify your clicks. Click. All right, yes and the answer is, C, excellent. Right? Because in fact, we just did that.

That won't happen on the actual exam. So here's the next question. Let the Hamiltonian on a dagger U_n equal e_n plus $\hbar \omega$ -- that should be an \hbar -- a dagger U_n . What can you say about a dagger U_n ? Here we should probably say, what can one say, because it's possible-- OK, what can one say?

Here the assumption-- just to say it out loud-- the assumption is that U_n is an eigenfunction of the energy operator h with eigenvalue e_n . You've got five seconds. All right, we are at 50/50. So discuss amongst yourselves.

All right. OK, go ahead and enter again. Enter your modified guesses. And you have 10 seconds to do so. This is a lot better.

OK, this is great. So this is one of those really satisfying moments. It's improved, but there's still some real doubt here. So I would like to get one person to argue for b and one person to argue for c. So who wants to volunteer for each of those?

AUDIENCE: I'll argue for c.

PROFESSOR: Who's going to argue for b? Someone's got to argue for b, come on.

AUDIENCE: You can argue.

PROFESSOR: I'm not going to argue for b. I'm not going to argue for c either. That defeats the

purpose.

I'm the professor. I have to say this all the time. So who's going to argue for b?

OK, you argue for b. Who's going to argue for c? All right, yeah, that worked out well. Great. Argue for b. You can do it.

AUDIENCE: So I originally accidentally mis-clicked b, so I guess I can do this. So originally I didn't read the question, and I thought since you were acting the letter operator on the eigenstate that you'd get a proportionality constant times some eigenstate. So that's why it could potentially be b. But I disbelieve.

PROFESSOR: It's not the best argument imaginable for b, but we'll take it. So thank you. When I put someone in an impossible position. So c.

AUDIENCE: All right. I argue for c. So we can see just from the first line here that this is clearly going to be a stationary [INAUDIBLE]. However, we can also see that it is a distinctly different energy from the state that you get an eigenfunction [INAUDIBLE]. So I said c, because it was not the same state.

PROFESSOR: Fantastic. That's exactly right. So let me walk through that. Let me say that allowed.

So from the first line, it is clear that the object U_n is an eigenfunction of the energy operator. And it's got eigenvalue e_n plus $\hbar \omega$. So it is a stationary state. It's an eigenfunction of the energy [INAUDIBLE].

However, it is not the same one, because it has a different eigenvalue. So what it means to be the same energy eigenstate is it has the same eigenvalue. If it has a different energy eigenvalue, it is a different state and they're orthogonal. We proved this before.

Two states with different energy are orthogonal. So not only are they not the same, they don't even have any overlap. So it is a stationary state, but it's not proportional with the state U_n . It's an important one. Questions about that one before we move on to the next?

AUDIENCE: How did you tell that it's proportional to the state U_n ?

PROFESSOR: If it were proportional to the state U_n , then it would be some constant times U_n . That's what we mean by saying it's proportional. But then if we acted on it with the energy operator, what would the eigenvalue be?

Let me say this again. Suppose I have a state the U_n , which I know that if I act with the energy operator on it-- or this is sometimes called $\hbar U_n$ is equal to $E_n U_n$. If I act with E on αU_n , where α 's a constant, what is this equal to? $\alpha E_n U_n$.

So the eigenvalue is the same. It's E_n , because it can divide through by [INAUDIBLE]. So if you have the same state-- meaning proportional to it-- then we'll have the same eigenvalue. But this manifestly has a different eigenvalue. Cool?

Awesome, OK other questions? Yeah.

AUDIENCE: I must have missed the-- this is [INAUDIBLE]. What is the [INAUDIBLE].

PROFESSOR: Ah, good. So H is another name. So some books use H and some books use E . And the reason goes back to classical mechanics.

There's an object called the Hamiltonian, which is sort of like saying-- I don't know. It's like saying bread and [NON ENGLISH SPEECH] are the same thing. But they evoke different things.

You say bread and you think like Dixie. And you say [NON ENGLISH SPEECH] and you think, the me of some gorgeous baguette. It has a different feel. It has a different feel.

So this evokes different things. In particular, to a classical mechanic the Hamiltonian is the generator of time translations. Now, we don't usually-- that wasn't the classic way of thinking about the energy operator in, say, Newton's time. But it is how we think about energy now.

So that's just terminology. For anyone who didn't get that, sorry about that. This is something that we've run across before, but I should emphasize.

So when you hear people say the Hamiltonian, that means the energy operator. The reason I say energy operator rather than saying Hamiltonian-- which is the more common thing in the Lingua Franca of quantum mechanics-- is because I want to impress upon you guys that it's just energy.

It is the energy, it's nothing else. It's not some strange beast. It's just energy. Other questions?

AUDIENCE: Which one's the bread?

PROFESSOR: Bread. Other questions? Next question. Which of these graphs shows the curvature of the wave function in a classically allowed region?

OK, five seconds left. All right, fantastic. And the answer is a, because the curvature has to be towards the axis.

In the classically allowed region, the wave function is sinusoidal. it's oscillatory. All right, and-- whoops. And the partner of that question is, what about in the classically disallowed region? Five seconds. OK, awesome, and basically everyone got this. B, great. Questions? Yeah, sorry.

AUDIENCE: [INAUDIBLE].

PROFESSOR: No. No, that's a very good question. So C and D, let me be a little bit more precise about it. It's a very reasonable question, so let me give you an answer.

Suppose you saw a wave function that had the curvature structure of C. It was curved oscillatory over there and curved exponential over there on the right. What would you say about the potential relative to the energy?

Say again? Yeah, good. So on one side we're allowed, and on the other side we're disallowed.

So if you saw a wave function that had that curvature structure, you'd immediately know that it was an allowed region on the left. But then on the right, it was classically disallowed, because it had curvature away from the axis. That cool?

So if you saw a wave function-- so for example, you might say look, you never get a wave function that does this. And that's not true. You could do that.

So this would be classically disallowed, because it's curvature down. This would be classically allowed. So the potential, for example, if this is the energy, the potential could be.

So a potential like this could lead to a function like this. So it's a very good question. I'm sorry, I should have seen what you were asking more immediately. Other questions following up on that?

No one ever asked that question before. And I just want to emphasize that you guys should just ask questions when you don't-- when you've confused about something, someone else is going to be confused about it. I wouldn't even necessarily know that the question is a meaningful question.

I get questions I've never heard before all the time in this class. Just don't hesitate to ask. It's always a good idea.

OK, next one. Wave function ψ has been expressed as a sum over energy eigenfunction U_n . Compared to the original wave function, is a function of x a set of coefficients, C_1 dot dot dot contains more or less, same information? Or it can't be determined? Or depends?

Five seconds. And the answer is.

AUDIENCE: C.

PROFESSOR: Same information. Good questions. Some of these you've seen before.

f and g are wave functions and c is a constant. Then the inner product of cf with g is equal to? Click away, 10 seconds. All right.

OK, you've got about five seconds. Still a lot of fluctuations here. Wow, I'm surprised by this one. Yeah, question.

AUDIENCE: Sorry, quick question on the previous one.

PROFESSOR: Oh, let's come back to the previous one after we do this one. Thanks. But remind me too after we move on.

So discuss, because about 1/3 of you got this one wrong. So discuss. So go ahead and put in your answers again. Good.

All right, another five seconds. OK, and the answer is. Good, get that complex conjugation right.

Little things like that can cause you an infinite amount of trouble. Especially as you'll notice on the problem set when you're computing time dependence of expectation values, getting that complex conjugation right is essential, because you need to see interference terms. And without the complex conjugation you will not get the right interference terms.

Oh yeah, a question from the previous problem. Thank you.

AUDIENCE: So since the summation with the constants, doesn't that technically give you the probability of each individual eigenstate?

PROFESSOR: The norm squared gives you the probabilities. But if you know the coefficients--- so this is a good question. So let me rephrase the question.

The question is, in the previous question we said, our wave function ψ of x is some function which we could draw. So I could represent this in two ways. I could either draw it, or I could represent it as equal the sum over n of $c_n \psi_n$ where ψ_n are the known energy functions. And the question is, did the c_n contain the same information?

So that was the question. And the point that's being asked about here is a very good question. The question is, look, we know what these c_n 's mean. What the c_n 's mean is the probability that you measure the energy-- given that the state of ψ , the probability [INAUDIBLE] energy E_n is equal to norm c_n squared.

And so the c_n 's tell you about probabilities. They don't tell you what the state is. They just tell you what a probability is that you'll measure a particular energy. Is that right?

So how can it contain the same information? And there are two important things to say about this. First off, this is of course true. And this does give you the probability. And the probability comes from the norm squared. So the phase of c_n doesn't particularly matter for that.

However the claim that I can expand this function in a basis of the energy eigenfunctions, in a basis of other functions, is a statement that if I know these c_n 's, then I take c_1 multiplied by ϕ_1 , c_2 multiplied by ϕ_2 , add them all back up, and I get nothing other than this original function. So if I know the c_n 's, I can just construct that original function explicitly. So this can't contain any more or any less information than the list of c_n 's, as long as I know what basis I'm talking about.

So here's the disturbing thing about that. When we say the wave function is the state of the system-- it contains all knowledge you could possibly have access to. And not just knowledge that you could have access to, it contains all of the information of the state. And then we look at this and say, well look, it doesn't give us something deterministic. It gives us something probabilistic.

It tells us the probability distribution for measuring the various different energies. What does that immediately tell you since they're equivalent information? This guy tells you nothing more than the probabilities. It doesn't tell you what energies you'll measure, it tells you the probabilities with which you will measure energies.

So they're the same thing. It's a very good question. And I think the way to take the sort of discomfort-- I don't know about you-- but that causes me is just to read that as saying, look, the wave function is giving you probabilistic information about measurements.

AUDIENCE:

I get that. It's just that if you have a wave function written as that or a wave function written as the [INAUDIBLE], I just see-- since it's written that way, then you know

what all the c_n 's are and what all the possible eigenstates are. And just [INAUDIBLE]. I understand they both contain the same information, but just written as each.

PROFESSOR: Excellent, excellent. So let me turn this around. Let me phrase this slightly different. Tell me if this is getting at the same point.

If I gave you a list of numbers, 1, 14, 12, does that tell you what function I'm talking about? No. I have to tell you that they're the expansion coefficients in, say, sine waves of a particular wavelength. Or energy eigenfunctions of some particular energy operator.

If I just give you a list of numbers, it doesn't tell you what function it is. I have to give you a list of numbers, and I have to tell you what basis I'm talking about. If you have the wave function, you don't need to know what basis you're talking about.

The wave function is basis independent. It's just the function. In a few minutes, I'm going to come back and subvert what I just said.

So the wave function is just what the wave function is. And if I give you a list of numbers, they're only equivalent when you also know the wave function. So in order for these two things to be equivalent, it's important that you know what bases are talking about. In particular, it's important that you know the energy eigenfunctions. Is that getting at the distinction you're making? OK, think more about it. And when the question becomes more sharp, ask. Other questions?

AUDIENCE: I think you kind of covered what my question was. But for the future, if the question is phrased such that if you know the c_n 's, you know all the information, are we supposed to just infer that if we know the c_n , we also know [INAUDIBLE]?

PROFESSOR: Yeah, usually. I try on problem sets in exams to make that distinction clear, but that will generally be assumed. Other questions?

I'm going to come back to this. I'm going to come back to this when we finish the clicker questions. Next question.

In the simple harmonic oscillator, the eigenvalues-- as we've described twice-- our E_n is $\hbar \omega$ times $n + 1/2$ for an integer n , and a measurement of energy will always observe one of these values. That's from postulate four. Three?

What can we say about the expectation value of the energy in some arbitrary state? This one isn't tricky, but it's not trivial. So think carefully about it.

You have five seconds. Enter your best guess. And you are at complete chance. I think we've just put you through a hardness box. So discuss.

OK go ahead and start putting in your modified response. You have another five seconds or so. OK, click.

OK, there's still a lot of disagreement here. So let's see. So who would like to argue for C? Who would like to argue for D? Go ahead.

AUDIENCE:

OK, so I'm going to argue for D. And against C. So C suggests that we would measure the value of the [INAUDIBLE] only for eigenstate that it addressed. But even if that changed, the stationary state consists of one and one eigenstate. So if we were to find the value, it can still be a combination of 1 and [INAUDIBLE]. And it doesn't have to fall to a certain eigenvalue.

PROFESSOR:

Excellent, excellent. So let me restate that. So the statement-- if I'm following your logic. Thank you.

So at the statement if I'm following your logic goes something like this. Look, if we had the wave function which was ψ_n times some phase, some ridiculous phase. Let's say the phase was 1. If we had the wave function as ψ_n , what will the expectation value of the energy operator be in this state?

E_n , right? Because it's the energy. It's a sum of all possible n of E_n times the probability that we measure E_n , by definition, but this is equal to sum over n of E_n times the expansion coefficient of our wave function in the energy eigenstate basis c_n squared.

Now, what's the expansion coefficient in our energy eigenbasis here? It's 1 for a particular value of n , and 0 for everything else. So what will our answer be? It will be 0 for everything except 1 for a particular value E_n corresponding to little m .

I should really call this m , so that dummy variables that we're summing over don't matter. So for this particular case the expectation value of e is equal to E_n .
Everyone cool with that?

So it certainly can be E_n . But is this the only way to get E_n ? No.

So for example, suppose we have $\alpha \phi_1$ plus $\beta \phi_3$. What will the answer be? Now first off, in order to make this normalizable, what does β have to be? It needs to be square of 1 minus α norm squared. Just so that they sum to 1 when norm squared.

So what I want to plot now the expectation value of the energy as a function of α . When α is 0, what is the expectation value of the energy? E_3 .

When α is 1-- which it can't be anything greater than 1-- when α is 1, what is the expectation value of the energy? E_1 . And how does the expectation value of the energy vary between these two as a function of α ?

Well, it's smooth. In particular, it's quadratic because it goes like the square of these coefficients. So it's something smooth that does this and has slope 0 at the end. And somewhere in between, it will hit E_2 .

But not only is this not equal to E_2 , it doesn't even contain E_2 as an element of the superposition. So this is a counter-example to C. This is an important point.

Now if I know that at some moment in time, the energy expectation value is this value, corresponding let's say to some particular α , and I let the system evolve in time, how does that expectation value evolve in time? How does the expectation value of the energy change over time?

It doesn't. The expectation value of energy is always time independent. Does the expectation value of position change over time? In general, yes.

When does it not change over time? In a stationary state. So working through this is what's on your previous problem set, as well as on your current problem set.

AUDIENCE: So is this anything like conservation of energy, where the expectation value of energy needs to stay the same if there's nothing going on?

PROFESSOR: Excellent, it sort of sounds like that. We'll make a sharp version of that later. So you should have that tinkling in the back of your head as like, oh look, we have a time independent potential, energy is conserved. That sounds kind of like the expectation value is constant.

But we need to make that more precise. So we'll find a precise version of that statement later in the semester. But it's a very good intuition to have. Other questions?

OK, so the answer is D. And you guys were about 4/5 on that at the end. Oops, that was not right.

OK, next one. This is a three-step question. We have observable A and B with eigenstates ψ_1 and ψ_2 , and ϕ_1 and ϕ_2 of A and B, and eigenvalues of a_1 , a_2 , and b_1 , b_2 . And the eigenstate's related in this linear fashion.

I measure observable A and get the value of a_1 . What's the state immediately after that measurement? What is the wave function? Five seconds. 2-1.

Click away. So there's pretty strong agreement among you all that the answer is A, and that is indeed the answer. Questions? So a slightly less obvious one.

Immediately after this measurement of A, observable B is measured. What's the probability the b_1 is found? Wow, there's some amazing initial transience there.

I could use the first couple of seconds as a random number generator. OK, that's it. And there's actually a third of you disagree with the correct answer. So I'm going to invite you to chat with each other.

Go ahead and update your answers. OK, five more seconds. One second. OK, nice.

Excellent. You guys went from about 65% to 90%, so that was great. The answer is E. It's the norm squared of the coefficient of eigenfunction corresponding to b_1 in the original state, which was ψ_1 .

Now imagine, on the other hand, that the grad student doing the measurement failed to in fact measure the observable B-- it was a bad day-- but instead measured A again. What's the probability that the second measurement will yield a_1 ? Trust me on this one, there's so many ways. They read the package wrong, they did the wrong measurement.

AUDIENCE: [INAUDIBLE].

PROFESSOR: Sorry, thank you, thank you, thank you. Very good question. He accidentally measured B instead of A.

AUDIENCE: [INAUDIBLE].

PROFESSOR: I mean A instead of B. A instead of B. See how easy it is to make that mistake? He accidentally measured A instead of B. I wish that was intentional.

OK, and so and we got almost total unanimity. The answer is B. Yes, question.

AUDIENCE: I was just going to ask how quickly after.

PROFESSOR: Great question. Let's just assume it's instantaneous. Maybe sloppy, but he's very quick. OK, next question.

A system is in a state which is a linear superposition of n equals 1 and n equals 2 energy eigenstates, blah. What's the probability that measurement of the energy will yield the first energy eigenvalue? Another five seconds, 2-3-4-5.

Great, you guys totally agree that the answer is C. Great. That's it for the clicker questions for today.

So here's the thing I want to emphasize. The first is that these were totally

conceptual. They did not involve any computations. And yet, they're not trivial.

Some of them got 50% consistently across the class. So it's not just the calculations that are hard. Just thinking through the basic premises, the basic postulates, is really essential at this point. It's more important than the computation. Yeah.

AUDIENCE: Are you going to post this?

PROFESSOR: I will post these. I will post these. Any questions remaining after this-- oh, shoot!

I missed out on a whole opportunity. I'll do it next time. Poo. Any other questions before we move on to lecture part?

AUDIENCE: [INAUDIBLE].

PROFESSOR: Oh, you totally have to wait. Yeah, you totally have to wait. Any other questions? Yeah.

AUDIENCE: Do you ever have a [INAUDIBLE] function that's odd? Or just continues down [INAUDIBLE].

PROFESSOR: Well, that's a very good question. So the question is, can you ever have a potential function that is odd? So for example-- you're actually asking two questions, so let me disentangle them.

So one question is, can I have an a potential function which is odd and linear? So it just ramps all the way down. So does that seem physical? Is the energy bounded from below?

Well, that's a not a terribly good criterion because it's not bounded from below for the Coulomb potential either. But in the case of the Coulomb potential, it's not bounded from below in a little tiny region. Not in a huge swath.

So we're going to need a better definition than this, but the short answer to this one is no. This is not good. On the other hand, it's not the oddness that's bad. It's that the energy is continuing to be unbounded from below.

So let me be more precise than saying it's not happy. What important for an observable? What's important for the operator corresponding to an observable in quantum mechanics? What property must that operator have?

It has to be Hermitian. And [INAUDIBLE] that implies real eigenvalues. It implies that there's a basis of its eigenfunctions, all these nice things. What we will find is that the Coulomb potential-- the hydrogen potential-- gives us a nice Hermitian energy operator. We'll prove that the energy operator self-adjoint on a problem set.

However, this guy will not. So we're going to have one that is not bounded from below. And so while there's a sense in which it's Hermitian, there's not a good sense. It's got problems. But that's a more technical detail.

However, there's a second part of your question, which is you could have been asking, look, can you have a potential that's odd but let's leave aside the diverging at infinity. So for example, could you have a potential that does this so that it's odd, but it's bounded?

And that's fine. Nothing wrong with that. So odd isn't a problem. The problem is certain ways of diverging.

AUDIENCE: So what would happen in the case where you did have some kind of linear potential? You had a uniform electric field and something was moving. Is the problem that that potential technically isn't a physical potential because it doesn't go away?

PROFESSOR: How would you ever build a linear potential, right? So the question here is, look, I know how to build a linear potential. Turn a uniform electric field, and an electric field is the gradient to the electrostatic potential. So that means that the electrostatic potential is linear.

And the potential energy of a charged particle with charge Q in an electrostatic potential is q times the electrostatic potential. So the potential energy for that charged particle is a linear function. So how do I build such a thing? Well, I take two capacitor plates, and I dump some charge on this one, and the opposite charge on

this guy, and I build up a linear potential.

Well yeah, exactly. So that's the problem. So can I make this linear over an arbitrarily large domain? No.

I need an arbitrarily large amount of charge, and I need to push them apart from each other arbitrarily far, which takes an arbitrarily large amount of work. That's not terribly physical, right? Your arms are only so big.

So at the end and the day, we're always going to discover that the pathologies that show up in a potential like this, like the Hamiltonian's not self-adjoint and compactly supported [INAUDIBLE] continuous functions, then OK. That's going to always be some mathematical version of, your arms are only so big. You cannot build an infinitely large apparatus like this.

So yes, you're exactly right. It is not physical. Is it a mathematical problem one could analyze? Yes.

One could write a dissertation on the singularity structure of these differential equations. But that dissertation would never be read by a physicist. Other questions? Yeah.

AUDIENCE: Talking about the raising and lowering number. Can you find the raising and lowering for any [INAUDIBLE] potential.

PROFESSOR: Excellent question. No. So here's what you can do.

Suppose I have a bunch of energy eigenstates. I have some potential. This is the potential.

It's a crazy potential. And it has some definite energy eigenvalues. As we've shown in 1D, or as I've mentioned and proved in office hours, we can't find degenerate eigenfunctions in a one-dimensional potential.

So they're not degenerate. They're spaced a finite distance apart for wave functions that are convergent to 0 and infinity for bound states. So we can always construct

an operator, which I will simply define in this way. A , and just for flourish I will put a dagger on it which does the following. A dagger is defined as the operator that maps ϕ_n to ϕ_{n+1} .

So what's the rule? It takes this state to this state, this state to this state, this state to this state. And since we can extend-- since these are a basis, then that tells us how this acts on an arbitrary function, because it acts on the superposition as acting on each term. So we can define it in this way, but here's the question.

What does it mean for the raising operator to be the raising operator? It wasn't that it lifted to raise the operator. That's not what started out the whole machinery. Where the raising operator got its juice was from this computational relation, a dagger with E was equal to $\hbar \omega$ a dagger.

So can we build an operator a dagger that raises states and commutes with the energy operator in this fashion? No, that depends on the energy operator. This is not a general property. This is not always something we can do.

If we could, then that energy operator would have evenly spaced energy eigenfunctions. On the other hand, it's not like it's never useful. So consider the following example. And I think this was on your problem set but I don't remember. So let me just say it.

If we write a dagger a is equal to n , then the commutation relation for this-- so the energy operator for the harmonic oscillator was equal to $\hbar \omega (n + 1/2)$. That's what we derived last time. If we define a dagger a as n , then the commutator of n with a dagger is a dagger. And the commutator of n with a is equal to minus a .

Suppose the energy operator is equal to $\hbar \omega (n + n^2)$. Does the raising operator commute with the energy operator to give you a dagger again? No, because it commutes to give you an a dagger and it commutes with this guy to give you, well, a constant times n^2 a dagger, which is a slightly funny [INAUDIBLE].

So this doesn't have the same computational relations. On the other hand, we know

how to build the functions of the n operator because they live in a tower by this algebra. By this commutator relation, they live in a tower.

So I can build the eigenfunctions of n . And then I can take those eigenfunctions of n and act on them with e and discover that they're also eigenfunctions of e . Because e acting on them is just n , and n cubed acting on the eigenfunctions of n . So the states with definite n are still eigenfunctions of the energy operator. But they're not evenly spaced.

Now can every potential be written in terms of this n operator. So I leave that you as an exercise. The answer is no. Try it.

AUDIENCE: [INAUDIBLE].

PROFESSOR: Yeah, exactly. So there's always something you could formally define in this fashion. It's not always useful to you, because it doesn't always commute with the energy operator in a nice way.

So then you can ask, are there properties that aren't nice about this? Well, there's something called the supersymmetric method in 1D quantum mechanics. But that's sort of beyond the scope. Come to my office hours and ask me this question again.

This is a miracle of the harmonic oscillator. Other questions?

AUDIENCE: So to construct the [INAUDIBLE] we use a condition that $a\psi_0 = 0$?

PROFESSOR: Yes.

AUDIENCE: So how did we come up with that? Did we demand that on physical grounds that it had to be 0?

PROFESSOR: Yes, we did. So excellent question. So the question is this. When we constructed using the operator method for the harmonic oscillator, when we constructed the ground state, we said the ground state, ψ_0 , is that state which is annihilated by the lowering operator. $a\psi_0 = 0$.

So where did that come from? Were we forcing that for physical reasons? For mathematical reasons? What was the reason?

So this was very important, so let's go through it slowly. So remember the step that came immediately before this step. Immediately before this step we said, look the expectation value of energy can always be written as a strictly positive thing. It's equal to the integral and you should derive this for yourself.

I didn't derive it, but this is a good thing to derive for yourself. It takes only a couple of steps. Integral overall momentum of ψ tilde of p -- the Fourier transformer norm squared. This is the probability that you have given the state ψ . This is the probability density that you have momentum p , p squared upon $2m$.

Plus the integral dx ψ of x norm squared $m\omega$ squared upon $2x$ squared. This is the probability that you're at position x , given the wave function ψ . And that's the value of the potential.

Now this is a strictly positive quantity, as is this. Strictly positive, as is this. This is a plus. This quantity must always be greater than or equal to 0. Correct?

However, we derive from the operator relations that if we take away function ψ sub e and we act on it with the lowering operator a , this defines a new state, which up to some normalization is an eigenfunction of the energy operator with energy e minus $\hbar\omega$. And we show this by showing that the energy eigenfunction acting on this gives exactly this coefficient times this back.

It's an eigenfunction. We just showed explicitly that it's an eigenfunction. So this suggest that if you have a state with energy e , you could also build a state with energy e minus $\hbar\omega$. And thus we can repeat e minus $2\hbar\omega$, and this turtles all the way down.

The problem is, for any finite value of e , eventually this tower will get negative. Here's 0. But the energy expectation value cannot be less than 0. It's got to be strictly greater than 0.

But if we were in this energy eigenstate and we measured the expectation value of energy, this is the value we would get, which would be negative. Whoops, that's a 2. So something is amiss.

How can this possibly make sense? And the answer up with was, well look, it's inescapable that acting with A on ψ gives us something that's an eigenfunction of the energy. However, it's possible that this function happens to be the 0 function. Not 0 energy, just 0, not a function.

Is zero normalizable? No, the integral of 0 squared is 0. No coefficient times that is a finite number, is 1.

So what must happen is it must be true that at some point we can't lower anymore. But we don't always act with a , so we always try to lower. The only out is if when we lower one particular guy, which I will call ψ_0 , the only out is if we try to lower ψ_0 , we don't get another state-- which we see we did here.

We don't get another state. Instead, we just get 0. We get no state. Not energy 0. 0, the function is 0, identically vanishing.

This does not describe the configuration of [INAUDIBLE]. The probability of it being anywhere is 0. That's bad.

So in order for the tower to end, in order for the energy to be bounded from below as the potential is, we need that there's a lowest state in the tower. Does that answer your question?

AUDIENCE: Yes.

PROFESSOR: Great. Question?

AUDIENCE: Are the energy eigenfunctions always strictly real?

PROFESSOR: Very good question. You're going to show this on a problem set if you haven't already. I thought you did, but maybe not. The energy eigenfunctions for potentials with only bound states can always be expressed as purely real or purely imaginary.

You can always decompose them in terms of purely real and purely imaginary.

Proving that will be a problem on one of your problem sets. But for the moment, let me just say, yes, you can always show that the energy eigenfunctions for 1D potential can always be expressed as purely real. I need that bound state condition, as we'll see later in the course. Yes.

AUDIENCE: So we lowered the energy [INAUDIBLE]. And the next energy is 0, right?

PROFESSOR: Yes.

AUDIENCE: Isn't that supposed to mean that the [INAUDIBLE] is $\hbar\omega$, not $\hbar\omega/2$?

PROFESSOR: Sorry? No, good. So what this says is given some state with energy e , when we lower it we get $\hbar\omega$ less. That doesn't tell you what e is. Other questions?

AUDIENCE: Just to expand on that. So if you get the 0 function, that doesn't actually correspond to any energy eigenvalue, let alone 0, right?

PROFESSOR: Correct. The function 0 is an eigenfunction of every operator. But it's a stupid eigenfunction of every operator.

In particular, it has nothing to do with the states we're interested in. We are interested in normalizable states, and that is not a normalizable state. You can't multiply it by any finite coefficient and get 1 when you square integrate it. One last question.

AUDIENCE: Maybe a stupid question. But does that mean that the only particle that can have 0 energy is the particle that doesn't exist?

PROFESSOR: Well, we need to add more work to that. So the question is, does that mean that the only particle that could really have 0 energy is some imaginary particle that doesn't exist. First off, imaginary particles may exist. We just haven't seen them.

AUDIENCE: [INAUDIBLE] real particle [INAUDIBLE] 0 energy.

PROFESSOR: Right. OK, so I need to say two things about this. So there's a technical complaint I

need to make about your question, which is a very fair question. But I need to make it all the same.

And the second is a more physical answer. So the technical complaint is, what do you mean by 0 energy? Do you ever measure energy directly? You measure energy differences.

So relative to what? We have to decide, relative to what? So for example, in the harmonic oscillator what we mean when we say the [INAUDIBLE] energy is non-zero is if we draw the potential, and we call the minimum of the potential $e = 0$, classical $e = 0$, then what we discover is that the energy of the quantum mechanical ground state is $\frac{1}{2} \hbar \omega$ above 0.

So is there any state of the harmonic oscillator with energy 0? No. If we have a free particle, is there a configuration with energy 0?

Well, that's an interesting question. So what are the eigenstates, e to the ikx . When the energy is 0-- so the energy is-- because it's just p^2 upon $2m$, which is $\hbar^2 k^2$ upon $2m$, the energy is $\hbar^2 k^2$ upon $2m$ for a free particle, no potential. And by 0 we mean the asymptotic energy equals 0.

Can you have a state with energy 0? This one's a little subtle, because if we take the energy 0, what must k be? 0, OK great.

So what's the wave function? Constant. Is that normalizable? Not so much. So can you put a single particle in a state with energy 0?

No, that would not seem to be possible, because that would be a non-normalizable state. Well you say, we deal with these exponentials all the time. We know how to deal with that. We don't use a single wave-- plane wave-- we use a wave packet with some width.

So I can build a wave packet with some average width, but will that average width ever have an expectation value of k equals 0? No, because it will always have contributions from k not equal 0 so that it's specially localized. So can you build a

normalizable state with energy 0, which is an eigenfunction of the energy 0? No, not in this case either.

So there's the technical complaint that you need to talk about what you mean relative to what. That was about your question. And the second of, can you ever have an energy eigenstate which is at the minimum of the potential? And that's also 0.

That's also no. You'll actually prove that on a later problem set. So with all that said, we have a mere-- that's awesome-- 15 minutes for a 14-page lecture. So can I do this fast?

No, I'm kidding. I won't do that to you. So I want to just-- instead of going through it. So this was not directly relevant for the exam. And this can come a little bit later.

So let me show you a couple of quick things. So the notes are posted online. Look at them. I'll go over them again in the future.

What do I want to do? So I want to introduce to you two ideas. One is Dirac notation. And this is hearkening back to basics of vector spaces.

And the second is, I want to tell you something about what the commutator means. So first off, let me ask you guys a question. I have one more clicker question for you.

All right, are you all ready? Everyone up with the clickers? Here's is the question.

[MUSIC PLAYING - JEOPARDY THEME]

OK, go ahead and start clicking now. Sigma's the uncertainty. Sorry, sigma is the uncertainty.

This is awesome. I swear the clicks are going in beat with the music. So let me give you three more seconds. 1-2-3.

This is awesome. You guys are totally at chance. You have even probability

distribution across the three.

So this is good, because we haven't actually introduced this idea yet in 804. So let me quickly talk you through this. So I'll leave it off.

So let me quickly talk you through this. And this is important. This is going to give you some intuition for what uncertainty means. And it's also going to give you some intuition for what the commutator means.

So this is a little more experience, a little more practice with operators. And we'll pick up on the Dirac notation and everything next time. Because I just want to get through this physics.

So consider two operators, A and B. So A and B are my two operators. They have hats. He has a top hat.

So we have-- you've got to make this stuff a little more light hearted. So we have two operators, A and B. And I want to ask, is it possible for there to be a function ϕ little a little b which is simultaneously an eigenfunction of a and an eigenfunction of b?

So now this is a pure math question. Given two operators, a and b, can you build a function which is simultaneously an eigenfunction of a and an eigenfunction of b? I will call this ϕ_{ab} .

Why not? OK, ϕ_{ab} has this property. And $a \phi_{ab}$ is equal to little a. Why not?

And $b \phi_{ab}$ is equal to little b on ϕ_{ab} . You can't stop me, I have now created this object.

From the existence of this state, what can you deduce about the operators a and b? And if you've already seen this before, that's cheating. So don't raise your hand. But if you haven't seen it before, just think through it. What does this tell you about the operators a and b?

AUDIENCE: [INAUDIBLE].

PROFESSOR: Great. That's a nice guess. I like that guess.

So let's check. So what would it mean for the commutator of a and b to be 0? So this is equal to ab minus ba .

And what the commutator does is it takes two operators and it gives you a new operator called bracket ab . And sometimes it's useful to call it c , for commutator. So it gives you a new operator. Given two operators you build a new operator.

And suppose now that we have a common eigenfunction of a and b . What does that tell you? Well, let's take the commutator-- a, b -- and let's take it and act on ϕ of ab on this common eigenstate.

So what is this equal to? This is equal to ab minus ba ϕ , which I can write as $ab \phi$ minus $ba \phi$ by linearity-- by addition, really. But b acting on ϕ gives me, by hypothesis, $\lambda \phi$. And then a acting on a constant times ϕ gives me a constant times that constant minus-- now here a acting on ϕ gives me $\lambda \phi$.

And then b acting on constant times ϕ gives me $\lambda \phi$. And now I can join these together. This is equal to $\lambda \phi$ minus $\lambda \phi$. But λ and λ are numbers.

If you take 7 times 5 and 5 times 7 and you subtract them, what do you get? 0. So in order for a and b to share a single eigenfunction ϕ , what must be true of their commutator?

It must have a non-zero kernel, exactly. The commutator must annihilate that particular common eigenfunction. Does it tell you that it kills every function? No.

But it must at the very least annihilate the shared eigenfunction. True? Consider two operators of the following form. Consider two operators whose commutator ab is equal to the identity times a constant.

Do these operators share any common eigenfunctions? They can't, because the

commutator-- the identity-- doesn't annihilate any wave functions. Any functions.

Nothing is 0 when acted upon by the identity. That's the definition of the identity.

The identity takes any function, gives it that function back. Cool?

Do a and b share any common eigenfunctions? Now let's think about what that means in terms of observables. Observables are represented by operators. What is the meaning of the eigenfunctions of those operators?

AUDIENCE: It corresponds to the [INAUDIBLE].

PROFESSOR: It corresponds to the state with a definite value of that observable. And the value is the corresponding eigenvalue. So the eigenfunction of an observable operator is a possible state with a definite value of that operator.

If I tell you that I have a state which is an energy eigenfunction, then that means it is a state with a definite value of the energy. If I measure that energy, I know exactly what I will get. Cool?

If a and b are two observables and their commutator is proportional to the identity, is it possible that there is a state with a definite value of a and a definite value of b simultaneously? No, because a state with a definite value of a would be an eigenfunction of a . And a state with a definite value of b would be an eigenfunction of b .

And there are no common states, no common eigenfunctions of a and b because the commutator never kills a state. It has no 0's. What is the commutator of x and p ? Commit this to memory, $i\hbar$.

Now when we write $i\hbar$, this is not really an operator. That's a number. What's the operator? The identity.

Are there any states which are simultaneously eigenfunctions of x -- the operator x -- and eigenfunctions of p ? Are there any states that have a definite position and a definite momentum simultaneously? Is that because we're ignorant?

Are we ignorant? Yes, OK good. So it is because the kinds of things that position [INAUDIBLE] forbid the existence of a state with a definite value of x and with a definite value of p simultaneously.

It is neither here, nor there, nor both, nor neither. So this tells us something very lovely. This tells us that if we're in a state with a definite value of x , what must be true of our uncertainty in the value of p ?

It can't be 0, because if our uncertainty in the knowledge p were 0, that would mean we were in a p eigenstate. We would have definite value of p .

So if a Δx is 0, Δp cannot be 0 for sure. Now how big does it have to be? Could it be arbitrarily small?

So here's a commutation relation. You will prove the following relation later on in the course. But I want to tell you now, because I want to give you some intuition for what the commutator is. What it means. What the physics of it is.

If I take two operators, a and b , which have a commutator-- which is a constant times the identity-- then the uncertainty in any state ψ of a times the uncertainty in the same state ψ of b must be greater than or equal to $1/2$ the absolute value of the expectation value of the commutator-- that's an amazing set of symbols-- of a with b . Square root.

AUDIENCE: You've got to take the square root.

PROFESSOR: Thank you.

AUDIENCE: Oh, I was kidding.

PROFESSOR: So the easier way to-- let's see a, b . Let's just make sure I'm getting this right. I'm the commutator of a and b . Oh no, no.

There's no square root. Sorry. Why are you messing with me?

Expectation value of a and b . Δa . Δp . \hbar .

Yes, good. So in particular, these expectation values should be taken with the state ψ . Sorry, it's late and I'm tired.

And this should be taken into state ψ . So we compute the commutator. We take the expectation value in our state. We take the norm, multiply it by $1/2$, and this is the bound. This is the claim.

So let's check this in the case of x and p . And do I have this using just math? This will come from just linear algebra.

So let's check this in the case of x and p . What's the commutator? 1 .

So what's the expectation value in a state, ψ , of $i\hbar$ times operator 1 ψ ? It's $i\hbar$. Great.

So what happens over here? What do we get for Δx in the state ψ times Δp in the state ψ ? This must be greater than or equal to $1/2$ times the absolute value of the expectation value of the commutator.

But the expectation value of the commutator is $i\hbar$. And the norm of that is just \hbar . This is the uncertainty relation.

So given some uncertainty in Δx , this tells us how much Δp is. Now, how does this right hand side depend on the wave function ψ ? It turns out it doesn't. It's explicitly independent of the particular wave function we're interested in.

For any wave function whatsoever, the uncertainty in x times the uncertainty in p must be greater than or equal to $1/2 \hbar$. And this is a consequence of the commutation, and more importantly, the failure of x and p to commute, because the ability to commute and get 0 is necessary for there to be common eigenstates.

Now imagine in this case a and b are that a and b that in fact do share a common eigenstate. So let's compute the uncertainty of a in the state ψ_{ab} , ψ_{ab} , ψ_{ab} , ψ_{ab} . And what is this equal to?

Well, what's the commutator acting on the state ψ_{ab} ? 0 . So this is greater

than or equal to 0.

And in fact, you can just check by going through the proof of this that it's just equal to 0. So what's the uncertainty of a ? Well, we know it's in the state with definite value of a . 0.

Uncertainty in b in this state, what's the uncertainty? 0. We know that it's in an eigenstate of b . So 0 is indeed greater than or equal to 0, so it satisfies the uncertainty relation.

Commuting is telling you about the possibility of the state having definite values of both operators simultaneously. And this is going to turn out to be enormously valuable when we talk about angular momentum, which is coming in a couple of weeks. OK, see you guys on Thursday here for the midterm.