Problem Set 8 — **Solutions**
8.04 Spring 2013 — **April 17, 2013**

Problem 1. (15 points) Superposition State of a Free Particle in 3D

(a) **(4 points)** Recall from lecture that the energy eigenstates of a free particle in 3D are given by

$$\psi = A\exp\left[i\left(\vec{k}\cdot\vec{r}\right)\right] = A\exp\left[i\left(k_x x + k_y y + k_z z\right)\right], \tag{1}$$

where A is some normalization constant. The energy is given by

$$E = \frac{\hbar^2|\vec{k}|^2}{2m} = \frac{\hbar^2(k_x^2+k_y^2+k_z^2)}{2m}. \tag{2}$$

In our case, we have

$$\psi(\vec{r},0) = \frac{\pi^{-\frac{3}{2}}}{2L^{3/2}}\sin(3x/L)e^{i(5y+z)/L} \tag{3a}$$

$$= \frac{\pi^{-\frac{3}{2}}}{4L^{3/2}}\left(e^{i3x/L} - e^{-i3x/L}\right)e^{i(5y+z)/L} \tag{3b}$$

$$= \frac{\pi^{-\frac{3}{2}}}{4L^{3/2}}\left[e^{i(3x+5y+z)/L} + e^{i(-3x+5y+z)/L}\right], \tag{3c}$$

which we can immediately see is the superposition of two energy eigenstates, one with $\vec{k} = (3,5,1)/L$ and the other with $\vec{k} = (-3,5,1)/L$. In both cases the energy is

$$E = \frac{\hbar^2|\vec{k}|^2}{2m} = \frac{35\hbar^2}{2mL^2}, \tag{4}$$

so this will be our result (with complete certainty) if we measure the energy at $t = 0$.

(b) **(4 points)** Since the momentum eigenstates are the same as the energy eigenstates for a free particle, Equation 3c can also be viewed as a superposition of momentum eigenstates. Using $\vec{p} = \hbar\vec{k}$, we can therefore say that the possible outcomes of a measurement of momentum are

$$\vec{p} = (3\hbar, 5\hbar, \hbar)/L \quad \text{and} \quad \vec{p} = (-3\hbar, 5\hbar, \hbar)/L. \tag{5}$$

Note that in this case the probability of finding the particle in one of these momentum eigenstates is *not* simply $|c_n|^2 = \frac{1}{16\pi^3L^3}$. This is because the wavefunction given in Equation 3c is not properly normalized (which also implies that the $|c_n|^2$'s don't sum to 1), and indeed *cannot* be normalized. However, since we know that only two outcomes are possible, and that they are equally likely (since their momentum eigenfunctions are multiplied by the same coefficient), we can say that the probability of measuring either of the momentum in Equation 5 is $\frac{1}{2}$.

(c) **(3 points)** We know from before that an energy eigenstate evolves by a multiplicative phase factor $e^{-iEt/\hbar}$. Since both energy eigenstates in our superposition state (Equation 3c) have the same energy, the time-dependent wavefunction is given by

$$\psi(\vec{r},t) = \frac{\pi^{-\frac{3}{2}}}{4L^{3/2}} \quad e^{i(3x+5y+z)/L} + e^{i(-3x+5y+z)/L} \quad e^{-iEt/\hbar} = \frac{\pi^{-\frac{3}{2}}}{2L^{3/2}} \sin(3x/L) e^{i(5y+z/L)} e^{-iEt/\hbar}, \tag{6}$$

where $E = 35\hbar^2/(2mL^2)$, as given in Equation 4.

(d) **(4 points)** Measuring $\vec{p} = (3\hbar, 5\hbar, \hbar)/L$ immediately collapses the wavefunction into the corresponding momentum eigenstate.

$$\psi(\vec{r},0) = A \exp\left[i\left(3x+5y+z\right)/L\right]. \tag{7}$$

Since this is also an energy eigenstate, the subsequent time evolution is once again given by

$$\psi(\vec{r},t) = A \exp\left[i\left(3x+5y+z\right)/L\right] e^{-iEt/\hbar}, \tag{8}$$

where again $E = 35\hbar^2/(2mL^2)$. We have left the normalization constant unspecified because the energy eigenstates of a free particle cannot be normalized.

Problem Set 8	**Solutions**
8.04 Spring 2013	**April 17, 2013**

Problem 2. (15 points) Degeneracies

(a) i. **(1 point)** The quantum number which determines the energy eigenvalue E is the wavenumber k. We see that in one dimension to a given energy E corresponds two wavenumbers $k = \pm\sqrt{\frac{2\text{mE}}{\hbar^2}}$, thus there are two linear independent sate for each energy eigenvalue.

ii. **(1 point)** The Hamiltonian of the free particle in one dimension is invariant under the parity operation, i.e. $x \to -x$. While the energy remains the same, the momentum operator transforms like the space coordinate: $\hat{p} \to -\hat{p}$.

(b) i. **(2 points)** The allowed energy eigenvalues are simply the sum of the energy eigenvalues of the two independent harmonic oscillators (since there is no interaction between them):

$$E_{n_x,n_y} = \hbar\omega_x \left(n_x + \frac{1}{2}\right) + \hbar\omega_y \left(n_y + \frac{1}{2}\right) = \hbar\omega \left(n_x + n_y + 1\right). \tag{9}$$

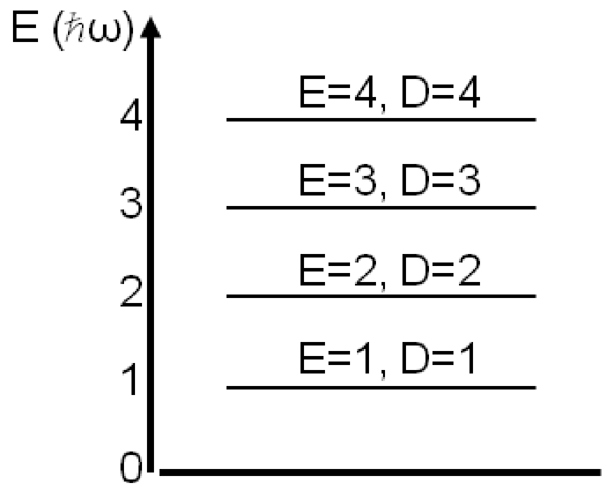
ii. **(2 points)** Let's make the notation $n = n_x + n_y$ so that the energy eigenvalue is $E_n = \hbar\omega\,(n+1)$. For a given eigenenergy E_n or principal quantum number n, n_x and n_y can take only the following values:

$$\begin{cases} n_x = 0, 1, \ldots, n-1, n \\ n_y = n, n-1, \ldots, 0, 1 \end{cases} . \tag{10}$$

So we see that there are $n+1$ ways to distribute the value of n among n_x and n_y. Thus the ground state $n = 0$ is non degenerate, first excited state $n = 1$ is double degenerate, second excited state $n = 2$ is triple degenerated, and the third excited sate $n = 3$ is four times degenerated.

iii. **(1 point)** Since the two independent oscillators have the same frequency, a rotation of the system around a axis perpendicular to the plane of the oscillators leaves the system unchanged, or in other words L_z component of angular momentum is conserved.

iv. **(2 points)** See the graph below (D stands for degeneracy):



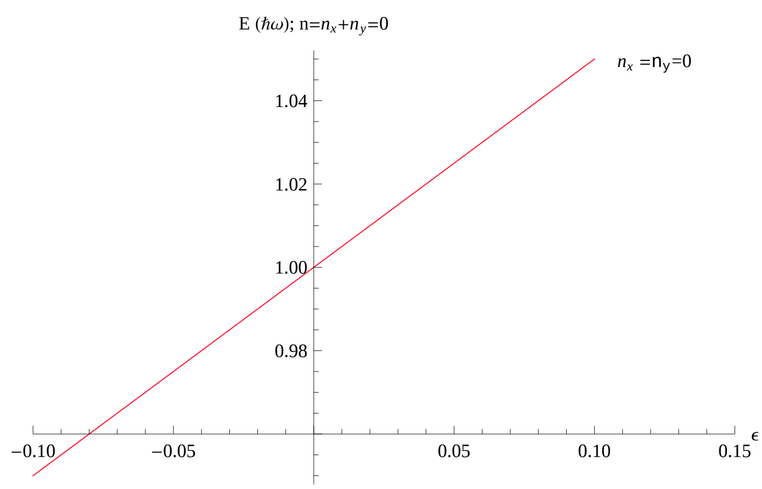
(c) i. **(1 point)** Since the two harmonic oscillators are still independent, the energy eigenvalue is:

$$E_{n_x,n_y}(\epsilon) = \hbar\omega_x\left(n_x+\frac{1}{2}\right)+\hbar\omega_y\left(n_y+\frac{1}{2}\right) = \hbar\omega\left(n_x+n_y+1\right)+\epsilon\hbar\omega\left(n_x+\frac{1}{2}\right). \tag{11}$$

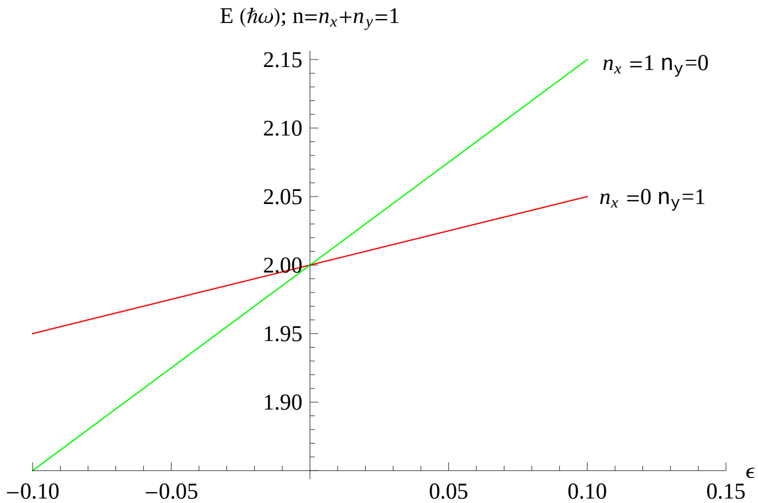
ii. **(1 point)** The system is no longer invariant under rotations, so the degeneracy is lifted.

iii. **(3 points)** Plotted below are the first ten eigenenergies in units of $\hbar\omega$ as function of ϵ (and the quantum numbers n_x and n_y):

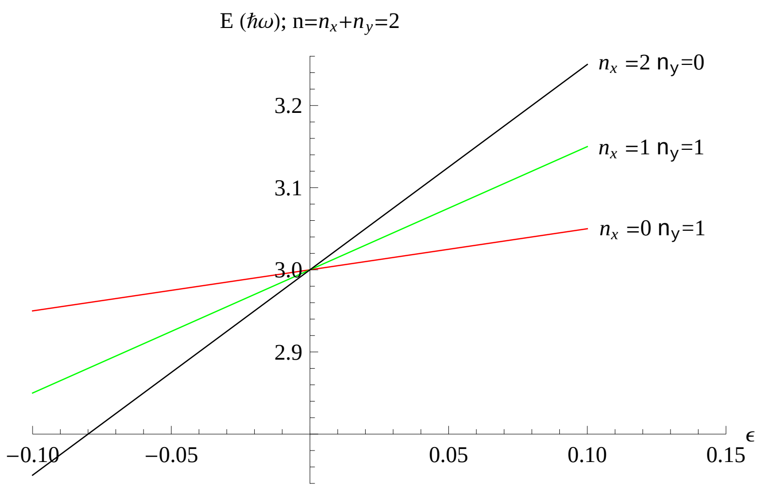
- $n = n_x + n_y = 0$



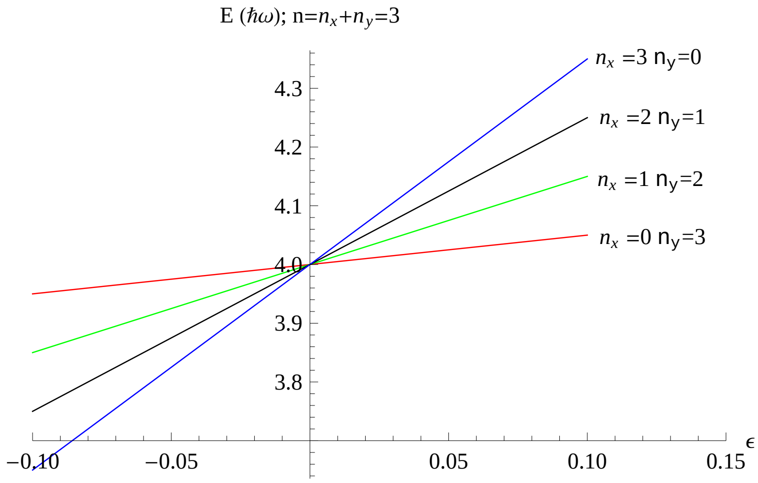
- $n = n_x + n_y = 1$



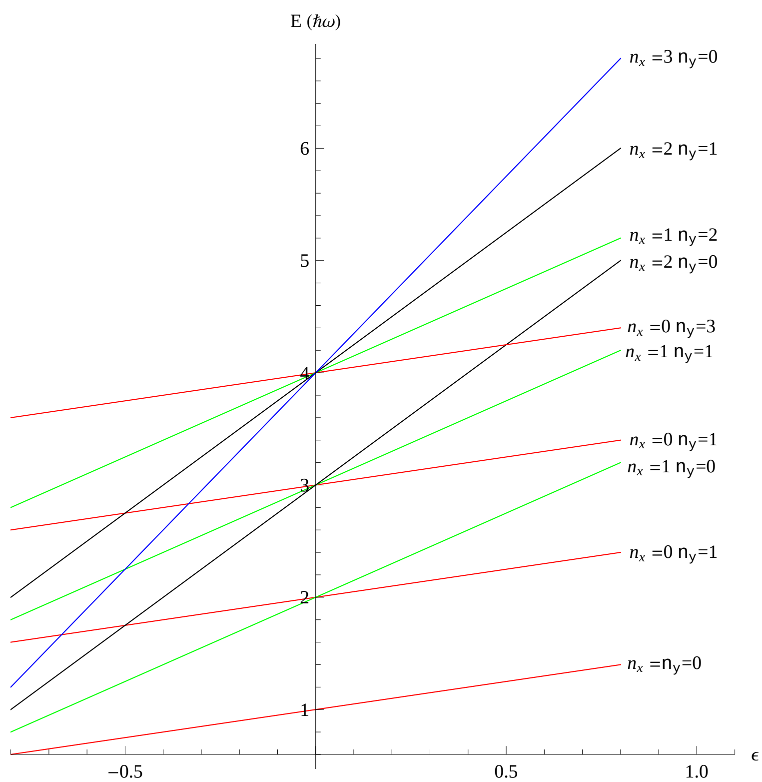
- $n = n_x + n_y = 2$



- $n = n_x + n_y = 3$



[Not required for full credit] In order to observe the crossings between various energy eigenvalues, below all ten eigenenergies are plotted on the same graph for a bigger range of ϵ. All the red eigenenergies correspond to $n_x = 0$, all the green eigenenergies correspond to $n_x = 1$, all the black eigenenergies correspond to $n_x = 2$, and the blue eigenenergy corresponds to $n_x = 3$.



iv. **(1 point)** As pointed earlier a degeneracy in energy eigenstates indicates that the system remains invariant under some symmetry transformation, or equivalently there is some conserved quantity. To each quantum number describing the conserved observable corresponds one energy eigenfunction, but since the energy eigenvalue does not depend on the conserved quantum numbers the energy eigenfunctions are degenerated.

Problem 3. (20 points) Mathematical Preliminaries: Angular Momentum Operators

(a) **(8 points)** First we deal with $[\hat{L}_y, \hat{L}_z]$:

$$[\hat{L}_y, \hat{L}_z] = [\hat{z}\hat{p}_x - \hat{x}\hat{p}_z, \hat{x}\hat{p}_y - \hat{y}\hat{p}_x] \tag{12a}$$
$$= (\hat{z}\hat{p}_x - \hat{x}\hat{p}_z)(\hat{x}\hat{p}_y - \hat{y}\hat{p}_x) - (\hat{x}\hat{p}_y - \hat{y}\hat{p}_x)(\hat{z}\hat{p}_x - \hat{x}\hat{p}_z) \tag{12b}$$
$$= \hat{z}\hat{p}_x\hat{x}\hat{p}_y - \hat{x}\hat{p}_z\hat{x}\hat{p}_y - \hat{z}\hat{p}_x\hat{y}\hat{p}_x + \hat{x}\hat{p}_z\hat{y}\hat{p}_x - \hat{x}\hat{p}_y\hat{z}\hat{p}_x + \hat{x}\hat{p}_y\hat{x}\hat{p}_z + \hat{y}\hat{p}_x\hat{z}\hat{p}_x - \hat{y}\hat{p}_x\hat{x}\hat{p}_z \tag{12c}$$
$$= (-\hat{x}\hat{p}_z\hat{x}\hat{p}_y + \hat{x}\hat{p}_y\hat{x}\hat{p}_z) + (\hat{y}\hat{p}_x\hat{z}\hat{p}_x - \hat{z}\hat{p}_x\hat{y}\hat{p}_x) + (\hat{z}\hat{p}_x\hat{x}\hat{p}_y - \hat{x}\hat{p}_y\hat{z}\hat{p}_x) + (\hat{x}\hat{p}_z\hat{y}\hat{p}_x - \hat{y}\hat{p}_x\hat{x}\hat{p}_z). \tag{12d}$$

The terms in the first two sets of parentheses are zero, because all the operators within a term commute with each other. For example,

$$-\hat{x}\hat{p}_z\hat{x}\hat{p}_y + \hat{x}\hat{p}_y\hat{x}\hat{p}_z = -\hat{x}^2\hat{p}_z\hat{p}_y + \hat{x}^2\hat{p}_y\hat{p}_z = -\hat{x}^2\hat{p}_z\hat{p}_y + \hat{x}^2\hat{p}_z\hat{p}_y = 0, \tag{13}$$

because $[\hat{x}_a, \hat{x}_b] = 0$, $[\hat{p}_a, \hat{p}_b] = 0$, and$[\hat{x}_a, \hat{p}_b] = i\hbar\delta_{ab}$. The remaining terms can be rewritten as follows:

$$\hat{z}\hat{p}_x\hat{x}\hat{p}_y - \hat{x}\hat{p}_y\hat{z}\hat{p}_x = \hat{z}\hat{p}_x\hat{x}\hat{p}_y - \hat{x}\hat{z}\hat{p}_y\hat{p}_x = \hat{z}\hat{p}_x\hat{x}\hat{p}_y - \hat{z}\hat{x}\hat{p}_x\hat{p}_y = -\hat{z}\underbrace{(\hat{x}\hat{p}_x - \hat{p}_x\hat{x})}_{=[\hat{x},\hat{p}_x]}\hat{p}_y = -i\hbar\hat{z}\hat{p}_y, \tag{14}$$

where we have made liberal use of the fact that the position and momentum commute unless they refer to the same coordinate (*i.e.* the same "x, y, or z"). Similarly,

$$\hat{x}\hat{p}_z\hat{y}\hat{p}_x - \hat{y}\hat{p}_x\hat{x}\hat{p}_z = (\hat{x}\hat{p}_x - \hat{p}_x\hat{x})\hat{p}_z\hat{y} = i\hbar\hat{p}_z\hat{y}. \tag{15}$$

Putting everything back together again, we get

$$[\hat{L}_y, \hat{L}_z] = i\hbar(\hat{y}\hat{p}_z - \hat{z}\hat{p}_y) = i\hbar\hat{L}_x. \tag{16}$$

One way to compute $[\hat{L}_z, \hat{L}_x]$ and $[\hat{L}_x, \hat{L}_y]$ would be to do the algebra explicitly, much like we did just now for $[\hat{L}_y, \hat{L}_z]$. An easier way, however, would be to make the following observation. If we look at the definition of the angular momentum operators,

$$\hat{L}_x = \hat{y}\hat{p}_z - \hat{z}\hat{p}_y, \quad \hat{L}_y = \hat{z}\hat{p}_x - \hat{x}\hat{p}_z, \quad \hat{L}_z = \hat{x}\hat{p}_y - \hat{y}\hat{p}_x \tag{17}$$

we see that making the replacements $x \to y$, $y \to z$, and $z \to x$ takes $\hat{L}_x \to \hat{L}_y$, $\hat{L}_y \to \hat{L}_z$, and $\hat{L}_z \to \hat{L}_x$. There is nothing mysterious about this, since x, y, and z are merely *labels* for the three coordinate axes, and we chose the replacements carefully so that the right-handedness of the coordinate system was preserved. Performing these replacements on Equation 16 yields

$$[\hat{L}_z, \hat{L}_x] = i\hbar\hat{L}_y \quad \text{and} \quad [\hat{L}_x, \hat{L}_y] = i\hbar\hat{L}_z. \tag{18}$$

(b) **(7 points)** We now consider the total angular momentum operator $\hat{L}^2 = \hat{L}_x^2 + \hat{L}_y^2 + \hat{L}_z^2$. The commutator of this operator with $\hat{L}_z$ is

$$[\hat{L}_z, \hat{L}^2] = [\hat{L}_z, \hat{L}_x^2 + \hat{L}_y^2 + \hat{L}_z^2] = [\hat{L}_z, \hat{L}_x^2] + [\hat{L}_z, \hat{L}_y^2] + [\hat{L}_z, \hat{L}_z^2] = [\hat{L}_z, \hat{L}_x^2] + [\hat{L}_z, \hat{L}_y^2]. \quad (19)$$

The first term can be written as

$$\begin{aligned} [\hat{L}_z, \hat{L}_x^2] &= \hat{L}_z\hat{L}_x^2 - \hat{L}_x^2\hat{L}_z = \hat{L}_z\hat{L}_x^2 - \hat{L}_x\hat{L}_z\hat{L}_x + \hat{L}_x\hat{L}_z\hat{L}_x - \hat{L}_x^2\hat{L}_z & (20a)\\ &= (\hat{L}_z\hat{L}_x - \hat{L}_x\hat{L}_z)\hat{L}_x + \hat{L}_x(\hat{L}_z\hat{L}_x - \hat{L}_x\hat{L}_z) & (20b)\\ &= [\hat{L}_z, \hat{L}_x]\hat{L}_x + \hat{L}_x[\hat{L}_z, \hat{L}_x] = i\hbar(\hat{L}_y\hat{L}_x + \hat{L}_x\hat{L}_y). & (20c) \end{aligned}$$

Similarly, we have

$$\begin{aligned} [\hat{L}_z, \hat{L}_y^2] &= \hat{L}_z\hat{L}_y^2 - \hat{L}_y^2\hat{L}_z = \hat{L}_z\hat{L}_y^2 - \hat{L}_y\hat{L}_z\hat{L}_y + \hat{L}_y\hat{L}_z\hat{L}_y - \hat{L}_y^2\hat{L}_z & (21a)\\ &= (\hat{L}_z\hat{L}_y - \hat{L}_y\hat{L}_z)\hat{L}_y + \hat{L}_y(\hat{L}_z\hat{L}_y - \hat{L}_y\hat{L}_z) & (21b)\\ &= [\hat{L}_z, \hat{L}_y]\hat{L}_y + \hat{L}_y[\hat{L}_z, \hat{L}_y] = -i\hbar(\hat{L}_y\hat{L}_x + \hat{L}_y\hat{L}_x). & (21c) \end{aligned}$$

The two terms cancel, so we get

$$[\hat{L}_z, \hat{L}^2] = 0. \quad (22)$$

As we argued above, x, y, and z are simply *labels* for the three coordinate axes. Since their labeling is arbitrary, and $\hat{L}^2$ is unaffected if we switch them, we can say without further calculation that

$$[\hat{L}_x, \hat{L}^2] = 0 \quad \text{and} \quad [\hat{L}_y, \hat{L}^2] = 0. \quad (23)$$

(c) **(5 points)** Let $\hat{L}_- = \hat{L}_x - i\hat{L}_y$ and $\hat{L}_+ = \hat{L}_x + i\hat{L}_y$. Then

$$[\hat{L}^2, \hat{L}_\pm] = [\hat{L}^2, \hat{L}_x \pm i\hat{L}_y] = [\hat{L}^2, \hat{L}_x] \pm i[\hat{L}^2, \hat{L}_y] = 0, \quad (24)$$

and

$$[\hat{L}_z, \hat{L}_\pm] = [\hat{L}_z, \hat{L}_x \pm i\hat{L}_y] = [\hat{L}_z, \hat{L}_x] \pm i[\hat{L}_z, \hat{L}_y] = i\hbar\hat{L}_y \pm \hbar\hat{L}_x = \pm\hbar\hat{L}_\pm. \quad (25)$$

Consider an eigenstate $\phi_{\ell,m}$ of both $\hat{L}_z$ and $\hat{L}^2$ such that

$$\hat{L}_z\phi_{\ell,m} = \hbar m\phi_{\ell,m}, \qquad \hat{L}^2\phi_{\ell,m} = \hbar^2 Q_\ell\phi_{\ell,m}. \quad (26)$$

The commutation relation (25) tells us that

$$\hat{L}_z\hat{L}_\pm\phi_{\ell,m} = \hat{L}_\pm\hat{L}_z\phi_{\ell,m} + [\hat{L}_z, \hat{L}_\pm]\phi_{\ell,m} = \hbar(m \pm 1)\hat{L}_\pm\phi_{\ell,m}, \quad (27)$$

thus $\hat{L}_\pm\phi_{\ell,m}$ is an eigenstate of $\hat{L}_z$ with eigenvalue $\hbar(m\pm1)$. Likewise, the commutation relation (24) tells us that

$$\hat{L}^2\hat{L}_\pm\phi_{\ell,m} = \hat{L}_\pm\hat{L}^2\phi_{\ell,m} + [\hat{L}^2, \hat{L}_\pm]\phi_{\ell,m} = \hbar^2 Q_\ell\hat{L}_\pm\phi_{\ell,m}, \quad (28)$$

from which we infer that $\hat{L}_\pm\phi_{\ell,m}$ is also an eigenstate of $\hat{L}^2$ and it has the same eigenvalue of $\phi_{\ell,m}$.

Problem 4. (15 points) Mathematical Preliminaries: Eigenfunctions of $\hat{L}^2$ and $\hat{L}_z$

(a) **(5 points)** For the angular parts of a wavefunction, the norm of a $\Psi(\theta, \phi)$ is given by

$$(\Psi|\Psi) = \int_0^{2\pi} d\phi \int_0^{\pi} \sin\theta d\theta |\Psi|^2. \tag{29}$$

Here we have

$$Y_{0,0} = \sqrt{\frac{1}{4\pi}} \quad \Rightarrow \quad |Y_{0,0}|^2 = \frac{1}{4\pi} \tag{30a}$$

$$Y_{1,0} = \sqrt{\frac{3}{4\pi}} \cos\theta \quad \Rightarrow \quad |Y_{1,0}|^2 = \frac{3}{4\pi} \cos^2\theta \tag{30b}$$

$$Y_{1,\pm1} = \mp\sqrt{\frac{3}{8\pi}} \sin\theta e^{\pm i\phi} \quad \Rightarrow \quad |Y_{1,\pm1}|^2 = \frac{3}{8\pi} \sin^2\theta \tag{30c}$$

So:

$$(Y_{0,0}|Y_{0,0}) = \int_0^{2\pi} d\phi \int_0^{\pi} \sin\theta d\theta |Y_{0,0}|^2 = \frac{1}{4\pi} \int_0^{2\pi} d\phi \int_0^{\pi} \sin\theta d\theta = 1 \tag{31a}$$

$$(Y_{1,0}|Y_{1,0}) = \int_0^{2\pi} d\phi \int_0^{\pi} \sin\theta d\theta |Y_{1,0}|^2 = \frac{3}{4\pi} \int_0^{2\pi} d\phi \int_0^{\pi} \sin\theta d\theta \cos^2\theta = 1 \tag{31b}$$

$$(Y_{1,\pm1}|Y_{1,\pm1}) = \int_0^{2\pi} d\phi \int_0^{\pi} \sin\theta d\theta |Y_{1,\pm1}|^2 = \frac{3}{8\pi} \int_0^{2\pi} d\phi \int_0^{\pi} \sin\theta d\theta \sin^2\theta = 1, \tag{31c}$$

where the last two integrals can be evaluated by making the substitution $u \equiv \cos\theta$. We now check for orthogonality:

$$(Y_{0,0}|Y_{1,0}) = \frac{\sqrt{3}}{4\pi} \int_0^{2\pi} d\phi \int_0^{\pi} \sin\theta d\theta \cos\theta = \frac{\sqrt{3}}{4} \int_0^{\pi} \sin 2\theta d\theta = 0 \tag{32a}$$

$$(Y_{0,0}|Y_{1,\pm1}) = \sqrt{\frac{3}{32\pi^2}} \int_0^{2\pi} d\phi \int_0^{\pi} \sin\theta d\theta \sin\theta e^{\pm i\phi} \tag{32b}$$

$$= \sqrt{\frac{3}{32\pi^2}} \underbrace{\int_0^{2\pi} e^{\pm i\phi} d\phi}_{=0} \int_0^{\pi} \sin^2\theta d\theta = 0 \tag{32c}$$

$$(Y_{1,0}|Y_{1,\pm1}) = \mp\frac{3}{\sqrt{32\pi^2}} \int_0^{2\pi} d\phi \int_0^{\pi} \sin\theta d\theta \cos\theta \sin\theta e^{\pm i\phi} \tag{32d}$$

$$= \mp\frac{3}{\sqrt{32\pi^2}} \underbrace{\int_0^{2\pi} e^{\pm i\phi} d\phi}_{=0} \int_0^{\pi} \sin^2\theta \cos\theta d\theta = 0, \tag{32e}$$

Note that we these three integrals we have exhausted all possible combinations. For example, it is unnecessary for us to check $(Y_{1,0}|Y_{0,0})$ because

$$(Y_{1,0}|Y_{0,0}) = \int_0^{2\pi} d\phi \int_0^{\pi} \sin\theta d\theta Y_{1,0}^* Y_{0,0} = \left(\int_0^{2\pi} d\phi \int_0^{\pi} \sin\theta d\theta Y_{0,0}^* Y_{1,0}\right)^* = (Y_{0,0}|Y_{1,0})^* = 0, \tag{33}$$

so if $(\Psi_A|\Psi_B) = 0$, then $(\Psi_B|\Psi_A)$ is automatically also zero.

(b) **(4 points)** If the spherical harmonics are eigenfunctions of $\hat{L}^2$ and $\hat{L}_z$, then they must satisfy the following eigenvalue equations:

$$\hat{L}^2\Psi = \lambda_{L^2}\Psi \qquad \Rightarrow \qquad -\hbar^2\left[\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right) + \frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\phi^2}\right]\Psi = \lambda_{L^2}\Psi \tag{34a}$$

$$\hat{L}_z\Psi = \lambda_{L_z}\Psi \qquad \Rightarrow \qquad -i\hbar\frac{\partial\Psi}{\partial\phi} = \lambda_{L_z}\Psi. \tag{34b}$$

Plugging the spherical harmonics into these equations, we see that they are indeed eigenfunctions of $\hat{L}^2$ and $\hat{L}_z$, with eigenvalues

- $Y_{0,0}$: $\lambda_{L^2} = 0$ and $\lambda_{L_z} = 0$.
- $Y_{1,0}$: $\lambda_{L^2} = 2\hbar^2$ and $\lambda_{L_z} = 0$.
- $Y_{1,\pm1}$: $\lambda_{L^2} = 2\hbar^2$ and $\lambda_{L_z} = \pm\hbar$.

(c) **(6 points)** From class we know that

$$\hat{L}_\pm = \hbar e^{\pm i\phi}(\partial_\theta \pm \cot\theta\partial_\phi). \tag{35}$$

We shall now follow the strategy learnt in class to obtain $Y_{l,l}$ and work out $Y_{42,-42}$. First, we require

$$Y_{42,-42} = P_{42,-42}(\theta)e^{-i42\phi}. \tag{36}$$

Imposing

$$\hat{L}_- Y_{42,-42} = \hbar e^{-i\phi}(\partial_\theta - 42\cot\theta)P_{42,-42}(\theta)e^{-i42\phi} = 0, \tag{37}$$

we find

$$Y_{42,-42} = c_{l,l}(\sin\theta)^{42}e^{-i42\phi}, \tag{38}$$

and thus,

$$Y_{42,-41} = \hat{L}_+ Y_{42,-42} \propto e^{i\phi}(\partial_\theta + \cot\theta\partial_\phi)(\sin\theta)^{42}e^{-i42\phi} \propto \sin^{41}\theta\cos\theta e^{-i41\phi}, \tag{39}$$

and the dependence on θ and ϕ is completely determined. To find the normalization factor $c_{42,-41}$, we impose

$$\int_0^{2\pi} d\phi \int_0^{\pi} d\theta \sin\theta |c_{42,-41}\sin^{41}\theta\cos\theta e^{-i41\phi}|^2 = 1. \tag{40}$$

Using Mathematica to work out the integral, we find

$$\int_0^{2\pi} d\phi \int_0^{\pi} d\theta \sin\theta |\sin^{41}\theta\cos\theta e^{-i41\phi}|^2 =$$

$$= 2\pi \int_0^\pi d\theta \sin^{83}\theta \cos^2\theta = 2\pi \frac{2^{40}}{3746069022360281995114 33275}, \qquad (41)$$

and thus

$$Y_{42,-41} = \sqrt{\frac{374606902236028199511433275}{2^{41}\pi}} \sin^{41}\theta \cos\theta e^{-i41\phi}. \qquad (42)$$

Problem 5. (20 points) Angular Momenta and Uncertainty

(a) **(5 points)** Since $\Psi \propto Y_{lm}$ is a normalized eigenstate of $\hat{L}^2$ and $\hat{L}_z$, we know that

$$\hat{L}^2\Psi = \hbar^2 l(l+1)\Psi \quad \text{and} \quad \hat{L}_z\Psi = \hbar m\Psi. \tag{43}$$

To find $\langle L_x \rangle$ and $\langle L_y \rangle$, we can make use of the operators $\hat{L}_+$ and $\hat{L}_-$, which are defined as

$$L_+ = L_x + iL_y \quad \text{and} \quad L_- = L_x - iL_y, \tag{44}$$

so

$$L_x = \frac{L_+ + L_-}{2} \quad \text{and} \quad L_y = \frac{L_+ - L_-}{2i}. \tag{45}$$

We therefore have

$$\langle L_x \rangle = \frac{1}{2}\int \Psi^*(L_+ + L_-)\Psi d\Omega \sim \int Y_{lm}^* L_+ Y_{lm} d\Omega + \int Y_{lm}^* L_- Y_{lm} d\Omega \tag{46a}$$

$$\sim \int Y_{lm}^* Y_{l,m+1} d\Omega + \int Y_{lm}^* Y_{l,m-1} d\Omega = 0, \tag{46b}$$

where in the last equality we have used the fact that the spherical harmonics are orthonormal, that is

$$(Y_{lm}|Y_{l'm'}) = \delta_{ll'}\delta mm', \tag{47}$$

so only if the l's and m's both match do we get a non-zero answer. Alternatively, we could have used notation that is somewhat more compact:

$$\langle L_x \rangle = \frac{1}{2}[(\Psi|L_+\Psi) + (\Psi|L_+\Psi)] \sim (Y_{lm}|L_+Y_{lm}) + (Y_{lm}|L_+Y_{lm}) \sim (Y_{lm}|Y_{l,m+1}) + (Y_{lm}|Y_{l,m-1}) = 0. \tag{48}$$

The steps in Equation 48 match those in Equations 46a and 46b exactly, and it is a good exercise to make sure you can translate between the different notations. For $\langle L_y \rangle$, we similarly have

$$\langle L_y \rangle = \frac{1}{2i}[(\Psi|L_+\Psi) - (\Psi|L_+\Psi)] \sim (Y_{lm}|L_+Y_{lm}) - (Y_{lm}|L_+Y_{lm}) \sim (Y_{lm}|Y_{l,m+1}) - (Y_{lm}|Y_{l,m-1}) = 0. \tag{49}$$

(b) **(6 points)** To find $\langle \hat{L}_x^2 \rangle$ and $\langle \hat{L}_y^2 \rangle$, we note that nothing we have done has broken the symmetry between the x and y axes. Our wavefunction is an eigenstate of $\hat{L}_z$, but nothing we have done so far has made a distinction between x and y. We can thus immediately say that $\langle \hat{L}_x^2 \rangle = \langle \hat{L}_y^2 \rangle$, and *in this case only* (it is not true for a general state):

$$\langle \hat{L}^2 \rangle = \langle \hat{L}_x^2 \rangle + \langle \hat{L}_y^2 \rangle + \langle \hat{L}_z^2 \rangle \quad \Rightarrow \quad \langle \hat{L}^2 \rangle = 2\langle \hat{L}_x^2 \rangle + \langle \hat{L}_z^2 \rangle \quad \Rightarrow \quad \langle \hat{L}_x^2 \rangle = \frac{\langle \hat{L}^2 \rangle - \langle \hat{L}_z^2 \rangle}{2}. \tag{50}$$

Thus,

$$\langle \hat{L}_x^2 \rangle = (\Psi|\hat{L}_x^2\Psi) = \frac{(\Psi|\hat{L}^2\Psi) - (\Psi|\hat{L}_z^2\Psi)}{2} = \frac{l(l+1)\hbar^2(\Psi|\Psi) - m^2\hbar^2(\Psi|\Psi)}{2} = \frac{\hbar^2}{2}[l(l+1)-m^2], \tag{51}$$

where the last equality followed from the fact that Ψ is normalized. In conclusion, then,

$$\langle \hat{L}_x^2 \rangle = \langle \hat{L}_y^2 \rangle = \frac{\hbar^2}{2}[l(l+1) - m^2]. \tag{52}$$

If you're uncomfortable with the symmetry argument, here's a different way to approach the problem. The operator $\hat{L}_x^2$ can be written as

$$\hat{L}_x^2 = \frac{1}{2}(\hat{L}_+ + \hat{L}_-)(\hat{L}_+ + \hat{L}_-) = \frac{1}{4}(\hat{L}_+^2 + \hat{L}_-\hat{L}_+ + \hat{L}_+\hat{L}_- + \hat{L}_-^2). \tag{53}$$

The first and last terms are of no consequence because when we "sandwich" them between Ψ's to find the expectation value we get zero from orthonomality:

$$(\Psi|\hat{L}_\pm^2\Psi) \propto (Y_{lm}|\hat{L}_\pm^2 Y_{lm}) \propto (Y_{lm}|Y_{l,m\pm 2}) = 0. \tag{54}$$

This means

$$(\Psi|\hat{L}_x^2\Psi) = \frac{1}{4}\quad (\Psi|\hat{L}_-\hat{L}_+\Psi) + (\Psi|\hat{L}_+\hat{L}_-\Psi) \tag{55}$$

We can deal with what's left by doing a little commutator algebra:

$$\begin{aligned} \hat{L}_\pm\hat{L}_\mp &= (\hat{L}_x \pm i\hat{L}_y)(\hat{L}_x \quad i\hat{L}_y) = \hat{L}_x^2 + \hat{L}_y^2 \pm i\hat{L}_y\hat{L}_x \quad i\hat{L}_x\hat{L}_y & \text{(56a)} \\ &= \hat{L}^2 - \hat{L}_z^2 \quad i[\hat{L}_x, \hat{L}_y] = \hat{L}^2 - \hat{L}_z^2 \pm \hbar\hat{L}_z, & \text{(56b)} \end{aligned}$$

where in the last equality we used the fact that $[\hat{L}_x, \hat{L}_y] = i\hbar\hat{L}_z$. Thus,

$$(\Psi|\hat{L}_\pm\hat{L}_\mp\Psi) = (\Psi|\hat{L}^2\Psi) - (\Psi|\hat{L}_z^2\Psi) \pm \hbar(\Psi|\hat{L}_z\Psi) = l(l+1)\hbar^2 - m^2\hbar^2 \pm m\hbar^2, \tag{57}$$

because Ψ is an eigenstate of both $\hat{L}^2$ and $\hat{L}_z$. Putting everything together, we get

$$\langle \hat{L}_x^2 \rangle = \frac{\hbar^2}{2}[l(l+1) - m^2], \tag{58}$$

just like before. A similar set of manipulations will give $\langle \hat{L}_y^2 \rangle$ and verify that $\langle \hat{L}_x^2 \rangle = \langle \hat{L}_y^2 \rangle$.

(c) **(4 points)** Since $\langle \hat{L}_x \rangle = 0$ and $\langle \hat{L}_x^2 \rangle = \langle \hat{L}_y^2 \rangle$, we have

$$\Delta L_x = \Delta L_y = \sqrt{\langle \hat{L}_y^2 \rangle - \langle \hat{L}_y \rangle^2} = \sqrt{\langle \hat{L}_y^2 \rangle} = \frac{\hbar}{\sqrt{2}}\sqrt{l(l+1) - m^2}. \tag{59}$$

The left hand side of the proposed uncertainty relation thus reads

$$\Delta L_x \Delta L_y = \frac{\hbar^2}{2}[l(l+1) - m^2]. \tag{60}$$

For a fixed l, this quantity is minimized when m is as large as possible *i.e.* when $m = l$. In such a situation, we have $\Delta L_x \Delta L_y = l\hbar^2/2$. As for the right hand side, we have

$$\frac{\hbar}{2}\langle L_z \rangle = \frac{m\hbar^2}{2} \tag{61}$$

This side is *maximized* precisely when the other side is *minimzed* (when $m = l$), and we get $\frac{\hbar}{2}\langle L_z \rangle = l\hbar^2/2$. In this case the two sides are equal, but for all other cases (where $m < l$) the left hand side is greater. We can therefore conclude

$$\Delta L_x \Delta L_y \geq \frac{\hbar}{2}\langle L_z \rangle. \tag{62}$$

It *is* possible for all components of angular momentum to vanish simultaneously. If a particle is in an eigenstate with $l = m = 0$, then from the relationships we have proved in this question we have $\langle \hat{L}_x^2 \rangle = \langle \hat{L}_y^2 \rangle = \langle \hat{L}_z^2 \rangle = 0$ as well as $\langle \hat{L}_x \rangle = \langle \hat{L}_y \rangle = \langle \hat{L}_z \rangle = 0$, so we can say $\Delta L_x = \Delta L_y = \Delta L_z = 0$.

Problem 6. (20 points) Lifting the Degeneracy of the Quantum Rigid Rotor

(a) **(4 points)** From the lecture notes we know that the spherical harmonics Y_{lm} are the eigenfunctions of the angular momentum operator squared $\vec{L}^2$:

$$\vec{L}^2 Y_{\text{lm}} = \hbar^2 l(l+1) Y_{\text{lm}}. \tag{63}$$

Thus the energy eigenvalues of the quantum rigid rotator are:

$$\hat{E} Y_{\text{lm}} = \frac{\vec{L}^2}{2I} Y_{\text{lm}} = \frac{\hbar^2 l(l+1)}{2I} Y_{\text{lm}} \Rightarrow E_{\text{lm}} = \frac{\hbar^2 l(l+1)}{2I}. \tag{64}$$

(b) **(4 points)** Although we labeled the energy eigenvalues (64) by both l (controlling the total angular momentum) and m (controling the z-component of the angular momentum) quantum numbers, in the case of the rigid rotator – and indeed for all spherically symmetric systems – the energy eigenvalues do not depend on m. From the lecture notes we know that for a given l the quantum number m runs from $+l$ to $-l$, so every energy eigenvalue is $2l+1$ degenerate.

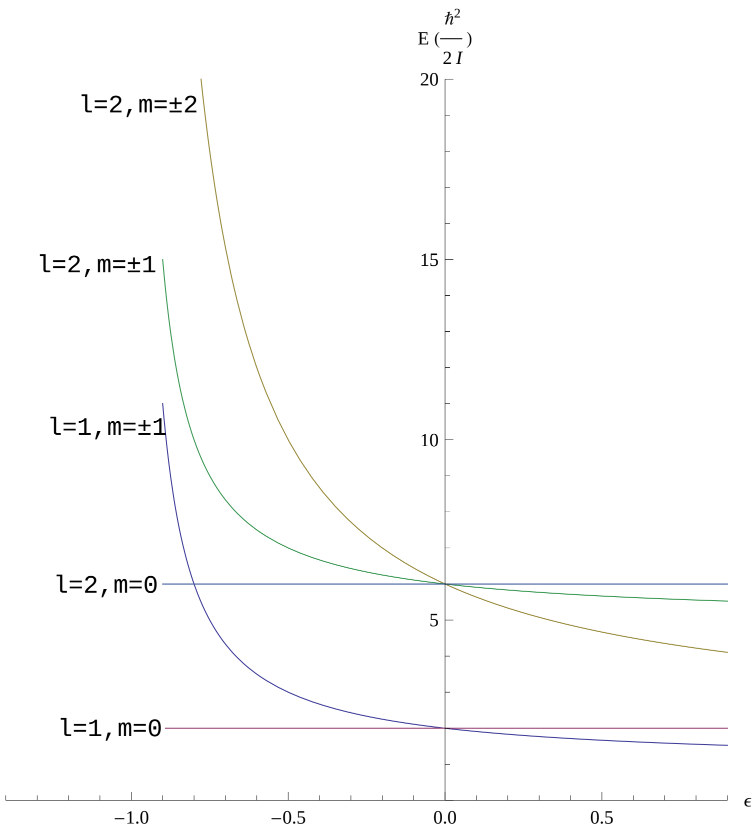
(c) **(4 points)** In this case, the energy operator is:

$$E = \frac{{L_x}^2 + {L_y}^2}{2I} + \frac{{L_z}^2}{2I(1+\epsilon)} = \frac{\vec{L}^2 - {L_z}^2}{2I} + \frac{{L_z}^2}{2I(1+\epsilon)} = \frac{\vec{L}^2}{2I} - \frac{\epsilon}{1+\epsilon}\frac{{L_z}^2}{2I}. \tag{65}$$

The spherical harmonics Y_{lm} are simultaneous eigenfunctions to both $\vec{L}^2$ and L_z, so the energy eigenfunctions of the above quantum energy operator are again the Y_{lm}:

$$\hat{E} Y_{\text{lm}} = \left(\frac{\vec{L}^2}{2I} - \frac{\epsilon}{1+\epsilon}\frac{{L_z}^2}{2I}\right) Y_{\text{lm}} = \left[\frac{\hbar^2 l(l+1)}{2I} - \frac{\epsilon}{1+\epsilon}\frac{\hbar^2 m^2}{2I}\right] Y_{\text{lm}}$$
$$\Rightarrow E_{\text{lm}} = \frac{\hbar^2 l(l+1)}{2I} - \frac{\epsilon}{1+\epsilon}\frac{\hbar^2 m^2}{2I}. \tag{66}$$

(d) **(5 points)** Let's take a concrete example, say $l = 1$ and $l = 2$:



The difference between two energy eigenstates $E_{l\ m\ }$ and E_{lm} is:

$$E_{l\ m\ } - E_{\mathrm{lm}} = \frac{\hbar^2[l\ (l\ +1) - l(l+1)]}{2I} + \frac{\epsilon}{1+\epsilon}\frac{\hbar^2\left(m^2 - m^{\ 2}\right)}{2I}. \qquad (67)$$

The first fraction on the right hand side of Eq. 67 is a constant and it does not affect the difference trend between the eigenenergies; the absolute value of the second fraction shows how close two eigenenergies can get since it depends on ϵ. We always can assume that $m^2 \geq m^{\ 2}$ (if not so exchange $E_{l\ m\ } \leftrightarrow E_{\mathrm{lm}}$) and the important term remaining is $\left|\frac{\epsilon}{1+\epsilon}\right|$. If $\epsilon \geq 0$ then $\left|\frac{\epsilon}{1+\epsilon}\right| = \frac{\epsilon}{1+\epsilon}$, but if $\epsilon \to -\epsilon \leq 0$ then $\left|\frac{\epsilon}{1+\epsilon}\right| = \frac{\epsilon}{1-\epsilon}$, so in the former case $\epsilon \geq 0$ the two energies are closer. Classically it means that for a given L_z as the momentum of inertia $I_z = I(1+\epsilon)$ increases the associated rotational energy decreases.

(e) **(3 points)** From Eq. 66 we see that, for a given l, the $\pm m$ eigenfunctions share the same eigenvalue, so the degeneracy is only partially lifted. This is because the system energy is still invariant under $L_z \to -L_z$, i.e. the rigid rotator starts to spin in the opposite direction but with the same angular momentum magnitude. The two opposite rotation directions correspond to $+m$ and $-m$.

We can break this invariance in many ways. For example, we might take $I_x \neq I_y$ – then we could tell whether we were spinning one way or the other by using the right hand rule with one finger along x and another along y. Unfortunately, the resulting energy eigenvalues cannot be solved for in closed form (this is directly related to the fact that the totally asymmetric rotor, $I_x = I_y = I_z$, is classically *chaotic*).

Instead, we can break this symmetry by adding to the energy operator the term,

$$E = \frac{\vec{L}^2}{2I} - \frac{\epsilon}{1+\epsilon}\frac{L_z^2}{2I} + qB_z L_z\,. \tag{68}$$

This term clearly breaks the symmetry $L_z \to -L_z$. Such a term could arise if we, say, rubbed Prof. Evans against a cat to charge him with some static electricity, then turned on a magnetic field $\vec{B} = B_z\hat{z}$ in the z direction. The resulting energy eigenvalues are then,

$$E_{\rm lm} = \frac{\hbar^2 l(l+1)}{2I} - \frac{\epsilon}{1+\epsilon}\frac{\hbar^2 m^2}{2I} + qB_z\hbar m \tag{69}$$

Thus breaking the symmetry again lifts the degeneracy.