Problem Set 2 **Solutions**
8.04 Spring 2013 **February 21, 2013**

Problem 1. (10 points) Wave-riding Mechanics

(a) **(4 points)** Given a dispersion relation $\omega(k)$, the phase velocity v_p is defined as

$$v_p \equiv \frac{\omega}{k}, \tag{1}$$

and is the rate at which a single plane wave (*i.e.* a single k-mode) moves within a wavepacket. The group velocity v_g, on the other hand, is defined as

$$v_g \equiv \frac{\partial \omega}{\partial k}, \tag{2}$$

and is the rate at which the envelope, or the "pattern" of a wavepacket moves[1]. Note that both are in general functions of k.

For gravity waves[2], the dispersion relation is given by

$$\omega = \sqrt{gk\tanh(kd)}. \tag{3}$$

The open ocean is deep, so $kd \approx \infty$, and $\tanh(kd) \approx 1$. This means $\omega \approx \sqrt{kg}$, and so

$$v_p \approx \sqrt{\frac{g}{k}} \tag{4a}$$

$$v_g \approx \frac{1}{2}\sqrt{\frac{g}{k}}. \tag{4b}$$

In deep water, the phase velocity is thus twice as quick as the group velocity.

(b) **(3 points)** From the dispersion relation (Equation 3), we see that as waves approach the beach (*i.e.* as d decreases), the angular velocity ω decreases. Furthermore, as one approaches the beach d gets small, so $\tanh(kd) \approx kd$ and $\omega \approx k\sqrt{gd}$. This gives $v_g \approx \sqrt{gd}$, which gets smaller as d gets smaller. Putting all this together, we see that the front of the envelope of the wavepacket slows down as the individual plane waves farther behind (*i.e.* still in deeper water) continue to move at a fairly quick rate. In other words, the difference between v_f and v_g changes. This causes the waves to "scrunch up" at the front of the envelope and the height of the waves to increase (since the total volume of water must remain constant). So the fact that waves appear to get taller as they approach the beach is *not* just an illusion. Eventually, the heights of the waves increase to such a point that nonlinearities set in, and the waves break.

[1] **Highly recommended** — To get a feel for the distinction between phase and group velocities, check out http://galileoandeinstein.physics.virginia.edu/more_stuff/Applets/wavepacket/wavepacket.html

[2] Not to be confused with *gravitational* waves, which are ripples in spacetime.

(c) **(3 points)** The surfer's impression is accurate. Individual plane waves advance "through" the envelope of a wavepacket quickly at rate of the phase velocity, but then die down in amplitude as they approach the front of the envelope in order to maintain the envelope's shape, which advances slowly at rate of the group velocity. To see an extreme case that illustrates this point, set the group velocity to zero in the applet cited in the footnote to part (a). One sees that at the nodes of the envelope, the individual plane waves die down completely.

From the applet and the discussion above, we know that individual plane waves interfere to reach their maximum amplitude in the mid-section of a wavepacket's envelope. As the wavepacket advances, it retains the shape of its envelope, so an observer standing at a fixed location experiences alternating groups of "big" (large amplitude) waves and "small" (small amplitude) waves.

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Problem 2. (10 points) Why I Don't Play Craps

(a) **(3 points)** Since a fair die is equally likely to give s from 1 to 12, one intuitively expects to get

$$\langle s \rangle = 6.5. \tag{5}$$

This is correct, but it pays to set things up a little more formally so that we can practice some of the techniques that will be useful when doing quantum problems. The definition of $\langle s \rangle$ is

$$\langle s \rangle = \sum_s s\,\mathbb{P}(s). \tag{6}$$

For a fair die, $\mathbb{P}(s)$ is equal[3] to $1/12$ for $s = 1, 2 \ldots 12$, so

$$\langle s \rangle = \frac{1}{8}\sum_{s=1}^{8} s = \frac{1+2+3+4+5+6+7+8+9+10+11+12}{12} = 6.5. \tag{7}$$

(b) **(3 points)** First we define Δs^2 to be

$$\Delta s^2 \equiv \langle (s - \langle s \rangle)^2 \rangle, \tag{8}$$

and Δs to be $\Delta s = \sqrt{\Delta s^2}$ (*i.e.* Δs^2 should be thought of as $(\Delta s)^2$ and *not* $\Delta(s^2)$). One way to tackle this problem would be to evaluate Δs^2 by brute force:

$$\Delta s^2 = \sum_s (s - \langle s \rangle)^2\,\mathbb{P}(s). \tag{9}$$

However, we saw in lecture[4] that

$$\Delta s^2 = \langle s^2 \rangle - \langle s \rangle^2. \tag{11}$$

[3] A more sophisticated way to write $\mathbb{P}(s)$ would be to say $\mathbb{P}(s) = \frac{1}{12}\sum_{i=1}^{12}\delta(s-i)$. Graphically, this would look like eight infinitely thin spikes centered on $s = 1, 2 \ldots 12$, which is just a way of saying that any given throw of the die will give some integer from 1 to 12 and nothing in-between. (The expectation value, being like an average, is allowed to take on non-integer values). Using this sort of formalism, the expectation values are integrals, which more closely resembles the sorts of expectation values that we deal with in quantum mechanics. For example, when finding $\langle s \rangle$, we have $\langle s \rangle \equiv \int_{-\infty}^{\infty} s\,\mathbb{P}(s)ds = \frac{1}{12}\int_{-\infty}^{\infty} s\left(\sum_{i=1}^{12}\delta(s-i)\right)ds = \frac{1}{12}\sum_{i=1}^{12}\left(\int_{-\infty}^{\infty} s\,\delta(s-i)ds\right) = \frac{1}{12}\sum_{i=1}^{12} i$, which is the same as Equation 6. Note that our $\mathbb{P}(s)$ satisfies the usual properties required for probability distribution functions, such as being positive everywhere and being properly normalized to one (*i.e.* $\int_{-\infty}^{\infty}\mathbb{P}(s)ds = 1$).

[4] As a reminder: $\Delta s^2 \equiv \langle (s - \langle s \rangle)^2 \rangle = \langle s^2 - 2s\langle s \rangle + \langle s \rangle^2 \rangle = \langle s^2 \rangle - 2\langle s\langle s \rangle\rangle + \langle\langle s \rangle^2\rangle$. If you're not convinced that $\langle a + b \rangle = \langle a \rangle + \langle b \rangle$, try writing out the expectation value integral explicitly. Now, unlike s, there is nothing random about $\langle s \rangle$ — it's simply a number. This means that any $\langle s \rangle$'s coming sailing out of the $\langle \ldots \rangle$'s, so that $\langle s\langle s \rangle\rangle = \langle s \rangle\langle s \rangle = \langle s \rangle^2$ and $\langle\langle s \rangle^2\rangle = \langle s \rangle^2$, giving

$$\Delta s^2 = \langle s^2 \rangle - 2\langle s \rangle^2 + \langle s \rangle^2 = \langle s^2 \rangle - \langle s \rangle^2 \quad \Rightarrow \quad \Delta s = \sqrt{\langle s^2 \rangle - \langle s \rangle^2}. \tag{10}$$

Thus, all that remains is for us to work out $\langle s^2 \rangle$, which we can do in a similar way to the manipulations in Equation 7:

$$\langle s^2 \rangle = \frac{1}{12} \sum_{s=1}^{12} s^2 = \frac{325}{6}. \tag{12}$$

Putting everything together gives

$$\Delta s = 3.45. \tag{13}$$

(c) **(4 points)** Let s_t represent the total number of spots shown by the two dice, and let s_1 and s_2 be the number of spots shown by the first and second die respectively. By definition, $s_t = s_1 + s_2$, so

$$\langle s_t \rangle = \langle s_1 + s_2 \rangle = \langle s_1 \rangle + \langle s_2 \rangle = 6.5 + 6.5 = 13. \tag{14}$$

Computing Δs_t is a little trickier. To avoid writing square root signs, let's consider Δs_t^2. We start by using the shortcut we found in Equation 10:

$$\begin{aligned} \Delta s_t^2 &= \langle s_t^2 \rangle - \langle s_t \rangle^2 = \langle (s_1 + s_2)^2 \rangle - (\langle s_1 \rangle + \langle s_2 \rangle)^2 \\ &= \langle s_1^2 + 2 s_1 s_2 + s_2^2 \rangle - \langle s_1 \rangle^2 - 2 \langle s_1 \rangle \langle s_2 \rangle - \langle s_2 \rangle^2 \\ &= \langle s_1^2 \rangle + 2 \langle s_1 s_2 \rangle + \langle s_2^2 \rangle - \langle s_1 \rangle^2 - 2 \langle s_1 \rangle \langle s_2 \rangle - \langle s_2 \rangle^2. \end{aligned} \tag{15}$$

Now, because the two die throws are *independent*, the *joint* probability $\mathbb{P}(s_1, s_2)$ of obtaining s_1 for the first die and s_2 for the second die is $\mathbb{P}(s_1)\mathbb{P}(s_2)$[5]. we have

$$\langle s_1 s_2 \rangle = \sum_{s_1, s_2} s_1 s_2 \, \mathbb{P}(s_1, s_2) = \sum_{s_1, s_2} s_1 s_2 \, \mathbb{P}(s_1)\mathbb{P}(s_2) = \left(\sum_{s_1} s_1 \, \mathbb{P}(s_1) \right) \left(\sum_{s_2} s_2 \, \mathbb{P}(s_2) \right) = \langle s_1 \rangle \langle s_2 \rangle, \tag{16}$$

which means our expression for Δs_t^2 reduces to

$$\Delta s_t^2 = \langle s_1^2 \rangle - \langle s_1 \rangle^2 + \langle s_1^2 \rangle - \langle s_2 \rangle^2 = 2 \Delta s^2, \tag{17}$$

where Δs refers to the spread defined for the single die throw analyzed in part (b). This gives

$$\Delta s_t = \sqrt{2 \Delta s^2} = 4.88. \tag{18}$$

[5] Please excuse the sloppy notation — $\mathbb{P}(s_1, s_2)$ is of course not the same *function* as $\mathbb{P}(s)$. After all, $\mathbb{P}(s_1, s_2)$ has two inputs while $\mathbb{P}(s)$ only has one.

Commentary: If we repeated this problem for N dice, we would have found

$$\langle s_t \rangle = N \langle s \rangle \tag{19a}$$
$$\Delta s_t = \sqrt{N \Delta s^2}, \tag{19b}$$

which means the *fractional* spread takes the form

$$\frac{\Delta s_t}{\langle s_t \rangle} = \frac{1}{\sqrt{N}} \frac{\Delta s}{\langle s \rangle}. \tag{20}$$

Now, suppose we forget about the dice and think about s as the result of some experimental measurement. Because of experimental uncertainties, the measurement comes with some error Δs, and $\Delta s / \langle s \rangle$ is the percentage error on the measurement. What Equation 20 tells us is that if we repeat the measurement over and over again and average our results, the percentage error will go down as $1/\sqrt{N}$.

Note that this behavior depends crucially on Equation 16, which was a result of each die throw/measurement being *independent*. Intuitively, what's happening is that with independent random errors, each individual measurement may be too high or too low, but on average the measurements will be high half the time and low the rest of the time. Thus, as more and more measurements are taken, the errors average down. On the other hand, if the measurements are off because of some *systematic* bias (*e.g.* if the measurements are always too high because of some instrumental miscalibration) then the errors do *not* average down. This is why physicists work so hard to eliminate sources of systematic error in their experiments.

Problem 3. (5 points) Dimensions of ψ

(a) **(2 points)** To determine the dimension of ψ we start with Born interpretation:

$$\mathrm{d}\mathbb{P}(x) = |\psi(x)|^2 dx\ , \tag{21}$$

i.e. the probability to find the particle in an infinitesimal interval dx around a position x is equal with the probability density $|\psi(x)|^2$ times the infinitesimal interval. Since a probability is a dimensionless quantity, $[\mathbb{P}] = 1$, we have:

$$1 = [\psi(x)]^2 \times L \qquad \Rightarrow \qquad [\psi(x)] = \frac{1}{\sqrt{L}} \tag{22}$$

(b) **(3 points)** As seen in lecture and proved below, we have:

$$\mathrm{d}\mathbb{P}(k) = \left|\tilde{\psi}(k)\right|^2 dk. \tag{23}$$

Recalling that k was defined as $k = \frac{2\pi}{\lambda}$, we see that $[k] = \frac{1}{L}$, so we can follow the same reasoning as above to get,

$$1 = \left[\tilde{\psi}(k)\right]^2 \times \frac{1}{L} \qquad \Rightarrow \qquad \left[\tilde{\psi}(k)\right] = \sqrt{L}. \tag{24}$$

Problem 4. (15 points) Fourier Transforms and Expectation Values

(a) **(5 points)** By definition of the expectation value of an operator, we have

$$\langle p \rangle = \int_{-\infty}^{\infty} \psi^*(x)\,\hat{p}\,\psi(x)\,dx, \tag{25}$$

where the hat on $\hat{p}$ emphasizes the fact that $\hat{p}$ is an *operator* and not a number. An immediate consequence of this is that except for in certain special cases, we can't swap the order of the terms in Equation 25, *i.e.* $\psi^*(x)\,\hat{p}\,\psi(x) \neq \hat{p}\,\psi^*(x)\,\psi(x)$. We can see this explicitly for this particular problem by plugging in the form of the momentum operator:

$$\langle p \rangle = \int_{-\infty}^{\infty} \psi^*(x)\,\frac{\hbar}{i}\frac{\partial}{\partial x}\,\psi(x)\,dx. \tag{26}$$

We can see that the derivative acts only on the second copy of ψ, whereas if we had written $\hat{p}\,\psi^*(x)\,\psi(x)$ it would've acted on the product of $\psi(x)$ and $\psi^*(x)$, which we know gives a different result from the product rule in calculus.

Let us now proceed by substituting into Equation 26 the definition of the Fourier transform:

$$\begin{aligned}
\langle p \rangle &= \int_{-\infty}^{\infty} \left(\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{iqx}\tilde{\psi}(q)dq\right)^* \frac{\hbar}{i}\frac{\partial}{\partial x}\left(\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{ikx}\tilde{\psi}(k)dk\right) dx &\text{(27a)}\\
&= \frac{1}{2\pi}\int_{-\infty}^{\infty}\left(\int_{-\infty}^{\infty} e^{-iqx}\tilde{\psi}^*(q)dq\right)\left(\int_{-\infty}^{\infty}\frac{\hbar}{i}\frac{\partial}{\partial x}\,e^{ikx}\tilde{\psi}(k)dk\right)dx &\text{(27b)}\\
&= \frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\tilde{\psi}^*(q)\tilde{\psi}(k)e^{-iqx}\frac{\hbar}{i}\frac{\partial}{\partial x}\,e^{ikx}dx\,dq\,dk &\text{(27c)}\\
&= \frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\tilde{\psi}^*(q)\tilde{\psi}(k)\left(\int_{-\infty}^{\infty}e^{-iqx}\frac{\hbar}{i}\frac{\partial}{\partial x}\,e^{ikx}dx\right)dq\,dk. &\text{(27d)}
\end{aligned}$$

A lot happened in the last few lines, and here are some subtleties to be aware of:

- In Equation 26, there are two copies of $\psi(x)$. When substituting these for their Fourier space representations, it is crucially important to use different variables for the Fourier variables (*i.e.* to *not* use k for both of them). To see this, recall that the integrals are really just fancy summations with q or k as dummy summation variables, and consider the following example, which serves as an analogy. Suppose we're trying to find the product of $a \equiv \sum_{i=1}^{2} i = 1+2 = 3$ and $b \equiv \sum_{j=1}^{2} j^2 = 1+4 = 5$. The answer is of course 15, but let's do this formally:

$$ab = \left(\sum_{i=1}^{2} i\right)\left(\sum_{j=1}^{2} j^2\right) = \sum_{i=1}^{2}\sum_{j=1}^{2} ij^2 = 1+4+2+8 = 15. \tag{28}$$

If we (incorrectly) use the same dummy index for the two summations, it is easy to forget that we're doing two sums, which leads to missing many of the terms in the sum:

$$ab = \left(\sum_{i=1}^{2} i\right)\left(\sum_{i=1}^{2} i^2\right) \rightarrow \sum_{i=1}^{2} i i^2 = 1 + 8 = 9 \quad \text{(Wrong!)} \tag{29}$$

where the right arrow signifies an incorrect logical step.

- In going from Equation 27a to 27b, we brought the momentum operator and the complex conjugate inside the Fourier integrals. This is allowed because differentiation and taking the complex conjugate of something are both linear operations acting on integrals, which are just sums. By definition, linear operations are ones where the same answer is obtained regardless of whether we sum (*i.e.* integrate) first and then "operate" or "operate" first and then sum.
- In going from Equation 27b to 27c, we switched the order of integration. This is allowed by Fubini's theorem (a purely mathematical result) thanks to the independence of our three integration variables x, q, and k.
- In going from Equation 27c to 27d, $\tilde{\psi}(k)$ passed right through the derivative, because it is a function of k and not of x.

Proceeding with the algebra from Equation 27d, we have

$$\begin{aligned}
\langle p \rangle &= \frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \tilde{\psi}^*(q)\tilde{\psi}(k)\left(\int_{-\infty}^{\infty} e^{-iqx}\hbar k\, e^{ikx} dx\right) dq\, dk && \text{(30a)}\\
&= \frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \tilde{\psi}^*(q)\tilde{\psi}(k)\,\hbar k \underbrace{\left(\int_{-\infty}^{\infty} e^{i(k-q)x} dx\right)}_{=2\pi\delta(k-q)} dq\, dk && \text{(30b)}\\
&= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \tilde{\psi}^*(q)\tilde{\psi}(k)\,\hbar k\,\delta(k-q) dq\, dk && \text{(30c)}\\
&= \int_{-\infty}^{\infty} \tilde{\psi}^*(k)\,\hbar k\,\tilde{\psi}(k)\, dk && \text{(30d)}\\
&= \int_{-\infty}^{\infty} |\tilde{\psi}(k)|^2\,\hbar k\, dk, && \text{(30e)}
\end{aligned}$$

which is our desired result. Note that Equation 30d is very similar in form to Equation 25, except we have k instead of x and $\tilde{\psi}$ instead of ψ. This suggests that while the momentum operator $\hat{p}$ takes the form of a derivative in real coordinate space, in Fourier space it is simply a multiplicative operator.

(b) **(5 points)** The steps to follow are the same as in part (a), except for the operator $\hat{p}$

which must be applied twice:

$$
\begin{aligned}
\langle \hat{p}^2 \rangle &= \int_{-\infty}^{\infty} \mathrm{dx}\psi^*(x) \left(-i\hbar\frac{\partial}{\partial x}\right)\left(-i\hbar\frac{\partial}{\partial x}\right)\psi(x) && \text{(31a)} \\
&= -\hbar^2 \int_{-\infty}^{\infty} \mathrm{dx}\psi^*(x)\frac{\partial^2}{\partial x^2}\psi(x) && \text{(31b)} \\
&= -\hbar^2 \int_{-\infty}^{\infty} \mathrm{dx} \left(\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} \mathrm{dq} e^{iqx}\tilde{\psi}(q)\right)^* \frac{\partial^2}{\partial x^2}\left(\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} \mathrm{dk} e^{ikx}\tilde{\psi}(k)\right) && \text{(31c)} \\
&= -\frac{\hbar^2}{2\pi}\int_{-\infty}^{\infty} \mathrm{dx} \left(\int_{-\infty}^{\infty} \mathrm{dq} e^{-iqx}\tilde{\psi}^*(q)\right)\left(\int_{-\infty}^{\infty} \mathrm{dk}\left(\frac{\partial^2}{\partial x^2}e^{ikx}\right)\tilde{\psi}(k)\right) && \text{(31d)} \\
&= -\frac{\hbar^2}{2\pi}\int_{-\infty}^{\infty} \mathrm{dx} \left(\int_{-\infty}^{\infty} \mathrm{dq} e^{-iqx}\tilde{\psi}^*(q)\right)\left(\int_{-\infty}^{\infty} \mathrm{dk}\left(-k^2\right)e^{ikx}\tilde{\psi}(k)\right) && \text{(31e)} \\
&= \hbar^2 \int_{-\infty}^{\infty} \mathrm{dq}\tilde{\psi}^*(q)\int_{-\infty}^{\infty} \mathrm{dk} k^2\tilde{\psi}(k)\underbrace{\frac{1}{2\pi}\int_{-\infty}^{\infty} \mathrm{dx}\, e^{i(k-q)x}}_{\delta(k-q)} && \text{(31f)} \\
&= \int_{-\infty}^{\infty} \mathrm{dk}|\tilde{\psi}(k)|^2(\hbar k)^2. && \text{(31g)}
\end{aligned}
$$

Again we see that the operator $\hat{p}$ becomes a multiplicative operator in the momentum space.

(c) **(5 points)** We wish to prove that

$$\langle f(\hat{p})\rangle = \int_{-\infty}^{\infty} |\tilde{\psi}(k)|^2 f(\hbar k) dk. \tag{32}$$

The left hand side can be Taylor expanded to give

$$
\begin{aligned}
\langle f(\hat{p})\rangle &= \langle f(0) + \hat{p}f'(0) + \frac{\hat{p}^2}{2!}f''(0) + \ldots\rangle && \text{(33a)} \\
&= f(0) + \langle\hat{p}\rangle f'(0) + \frac{\langle\hat{p}^2\rangle}{2!}f''(0) + \ldots && \text{(33b)}
\end{aligned}
$$

To evaluate this, we need to know how to work out expectation values of powers of $\hat{p}$, *i.e.* $\langle\hat{p}^n\rangle$. One way to do this would be to go all the way back to $\hat{p} = \frac{\hbar}{i}\frac{\partial}{\partial x}$ and to find $\langle\hat{p}^n\rangle$ by brute force, but an easier way would be to use our result from the previous part. There, we showed that in Fourier space, the momentum operator takes the simple form of multiplication by $\hbar k$. It follows, then, that powers of $\hat{p}$ correspond to powers of $\hbar k$ in Fourier space, which means

$$
\begin{aligned}
\langle f(\hat{p})\rangle &= f(0) + f'(0)\int_{-\infty}^{\infty} |\tilde{\psi}(k)|^2\,\hbar k\, dk + \frac{1}{2!}f''(0)\int_{-\infty}^{\infty} |\tilde{\psi}(k)|^2\,(\hbar k)^2\, dk + \ldots && \text{(34a)} \\
&= \int_{-\infty}^{\infty} |\tilde{\psi}(k)|^2\left(f(0) + \hbar k f'(0) + \frac{(\hbar k)^2}{2!}f''(0) + \ldots\right) dk && \text{(34b)} \\
&= \int_{-\infty}^{\infty} |\tilde{\psi}(k)|^2\, f(\hbar k)\, dk. && \text{(34c)}
\end{aligned}
$$

This is the same as Equation 32, so our proof is complete. Note that we implicitly used the fact that $\int_{-\infty}^{\infty} |\tilde{\psi}(k)|^2 dk = 1$. This can be seen either as a mathematical statement (Parseval's theorem from Fourier analysis), or a physical one (probabilities need to add up to 1 whether we work in real space or Fourier space).

Problem 5. (15 points) Delta Functions

(a) i. **(1 point)**

$$\int_{-3}^{1} \delta(x+2)\ (x^3 - 3x^2 + 2x - 1)\, dx = (-2)^3 - 3(-2)^2 + 2(-2) - 1 = -25. \tag{35}$$

ii. **(1 point)**

$$\int_{0}^{\infty} \delta(x-\pi)\ (\cos(3x) + 2)\, dx = \cos(3\pi) + 2 = 1. \tag{36}$$

iii. **(1 points)**

$$\int_{-1}^{1} \delta(x-2)\ e^{|x|+3}\, dx = 0, \tag{37}$$

because $x = 2$ is not in the range $[-1, 1]$.

(b) i. **(1 point)**

$$\int_{-\infty}^{\infty} x\,\delta(x) f(x)\, dx = 0f(0) = 0 = \int_{-\infty}^{\infty} 0f(x)\, dx \quad \Rightarrow \quad x\delta(x) = 0. \tag{38}$$

ii. **(1 point)**

$$\int_{-\infty}^{\infty} \delta(-x) f(x)\, dx = f(0) = \int_{-\infty}^{\infty} \delta(x) f(x)\, dx \quad \Rightarrow \quad \delta(-x) = \delta(x). \tag{39}$$

iii. **(1 point)** Assume $c > 0$, and let $u \equiv cx$. Then

$$\int_{-\infty}^{\infty} \delta(cx) f(x)\, dx = \int_{-\infty}^{\infty} \delta(u) f\left(\frac{u}{c}\right) \frac{du}{c} = \frac{1}{c} f(0) = \int_{-\infty}^{\infty} \frac{\delta(x)}{c} f(x)\, dx. \tag{40}$$

If $c < 0$, then the limits of integration get swapped during the substitution, so

$$\int_{-\infty}^{\infty} \delta(cx) f(x)\, dx = -\int_{-\infty}^{\infty} \delta(u) f\left(\frac{u}{c}\right) \frac{du}{c} = -\frac{1}{c} f(0) = \int_{-\infty}^{\infty} \frac{\delta(x)}{-c} f(x)\, dx. \tag{41}$$

Putting these two results together gives

$$\delta(cx) = \frac{1}{|c|}\delta x. \tag{42}$$

iv. **(1 point)** This can be shown directly simply by letting $f(x) \equiv \delta(x)$:

$$\int_{-\infty}^{\infty} \delta(a-x) f(x-b)\, dx = f(a-b) = \delta(a-b). \tag{43}$$

v. **(1 point)**

$$\begin{aligned}\int_{-\infty}^{\infty} \delta(x-a) f(x) g(x)\, dx &= f(a) g(a) = \int_{-\infty}^{\infty} \delta(x-a) f(a) g(x)\, dx && \text{(44a)}\\ \Rightarrow \quad f(x)\delta(x-a) &= f(a)\delta(x-a). && \text{(44b)}\end{aligned}$$

(c) For each of the proposed forms of the delta function, we essentially need to show that

$$\int_{-\infty}^{\infty} f(y)\delta(y-x)\, dy = f(x). \tag{45}$$

i. **(4 points)** Consider the inverse Fourier transform equation:

$$f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{ikx} \tilde{f}(k) dk. \tag{46}$$

Now suppose we substitute into this the formula for the Fourier transform, that is,

$$\tilde{f}(k) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-ikx} f(x) dx. \tag{47}$$

The result is

$$f(x) = \int_{-\infty}^{\infty} e^{ikx} \left(\frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-iky} f(y) dy \right) dk. \tag{48}$$

(Note the crucial renaming of the dummy variable in Equation 47 from x to y. See the first bullet point in the Problem 3 solutions for more details). Changing the order of integration gives

$$f(x) = \int_{-\infty}^{\infty} f(y) \underbrace{\left(\frac{1}{2\pi} \int_{-\infty}^{\infty} e^{ik(x-y)} dk \right)}_{\delta(x-y)} dy, \tag{49}$$

where we identified the term in the parentheses as a delta function by comparing our expression to Equation 45 (note that $\delta(x-y) = \delta(y-x)$ from Part b ii above). We can thus conclude that

$$\delta(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{ikx} dk. \tag{50}$$

ii. **(4 points)** We want to show that

$$\int_{-\infty}^{\infty} f(x) \left(\lim_{a\to 0} \frac{1}{a\sqrt{\pi}} e^{-x^2/a^2} \right) dx = f(0). \tag{51}$$

To do so, we first take the limit outside the integral:

$$\lim_{a\to 0} \int_{-\infty}^{\infty} f(x) \frac{1}{a\sqrt{\pi}} e^{-x^2/a^2} dx. \tag{52}$$

We now non-dimensionalize by letting $u = x/a$:

$$\lim_{a\to 0}\int_{-\infty}^{\infty} f(ua)\frac{1}{\sqrt{\pi}}e^{-u^2}du. \tag{53}$$

The only remaining factor of a resides in $f(ua)$, so we can simply take the limit and say $f(ua) \to f(0)$, which then comes out of the integral because it is now a constant:

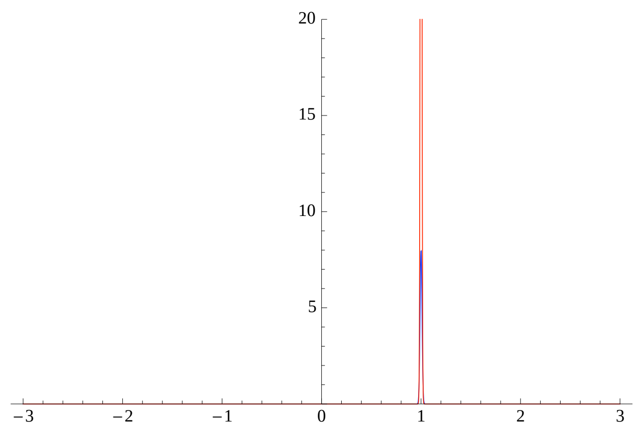
$$f(0)\lim_{a\to 0}\int_{-\infty}^{\infty} \frac{1}{\sqrt{\pi}}e^{-u^2}du = f(0), \tag{54}$$

where we have recognized that the integral is a standard Gaussian integral that evaluates to 1. We have thus shown that Equation 51 is true and $\delta(x) = \lim_{a\to 0}\frac{1}{a\sqrt{\pi}}e^{-x^2/a^2}$.

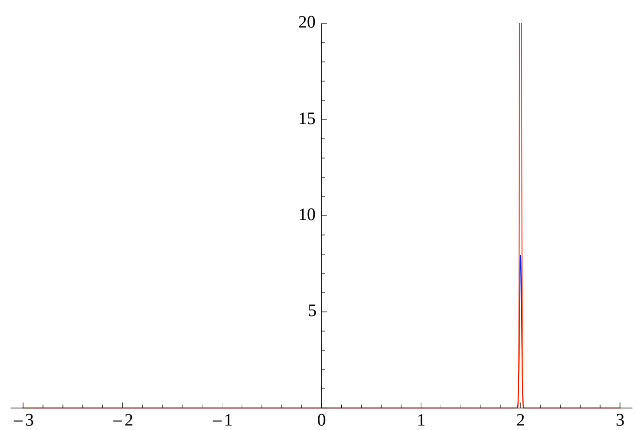
Problem 6. (30 points) Qualitative Structure of Wavefunctions

(a) **(8 points)** Because these are the official electronic solutions, we have chosen to plot these wavefunctions rather than sketch them. It is worth noting, however, that being able to sketch a function in a qualitatively accurate way is an important skill, so definitely get some practice if you're uncomfortable with it. (Remember, you won't get to use your computer on exams!). In the plots below, we sketch the real part of the wavefunction $\psi(x)$ in blue and the corresponding probability distribution $|\psi(x)|^2 = \psi^*(x)\psi(x)$ in red.

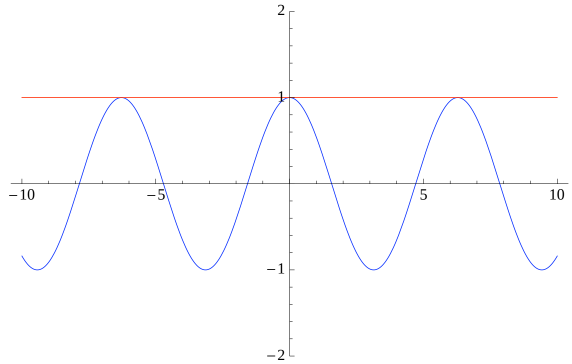
- $\psi_1(x) = \delta(x-1)$. Delta functions are infinitely narrow and infinitely tall, so they can't really be plotted. Shown below is a Gaussian approximation (do Problem 6 c to see why this is sensible!).



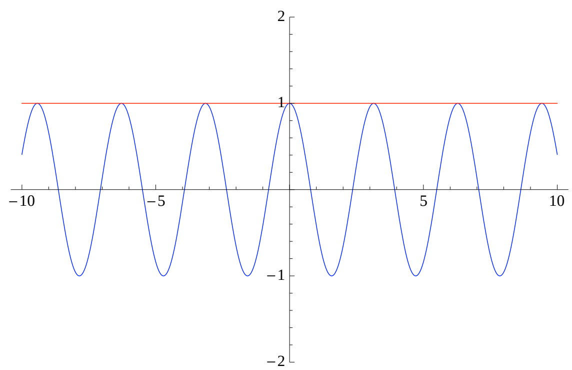
- $\psi_2(x) = \delta(x-2)$.



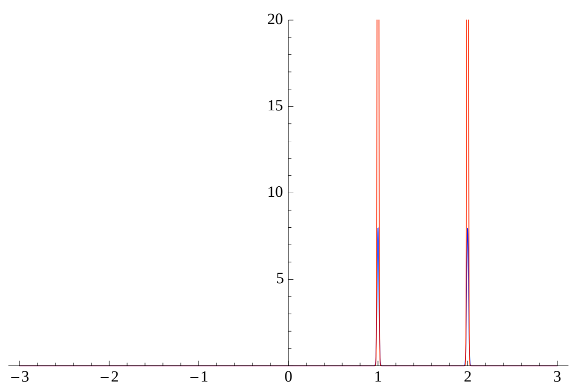
- Real part of the wavefunction: $\mathrm{Re}(\psi_3(x)) = \mathrm{Re}(e^{ix}) = \mathrm{Re}(\cos x + i \sin x) = \cos x$.
  Probability distribution: $|\psi_3(x)|^2 = \psi_3^*(x)\psi_3(x) = e^{-ix}e^{ix} = 1$.



- Real part of the wavefunction: $\mathrm{Re}(\psi_4(x)) = \mathrm{Re}(e^{i2x}) = \mathrm{Re}(\cos 2x + i \sin 2x) = \cos 2x$.
  Probability distribution: $|\psi_4(x)|^2 = \psi_4^*(x)\psi_4(x) = e^{-i2x}e^{i2x} = 1$

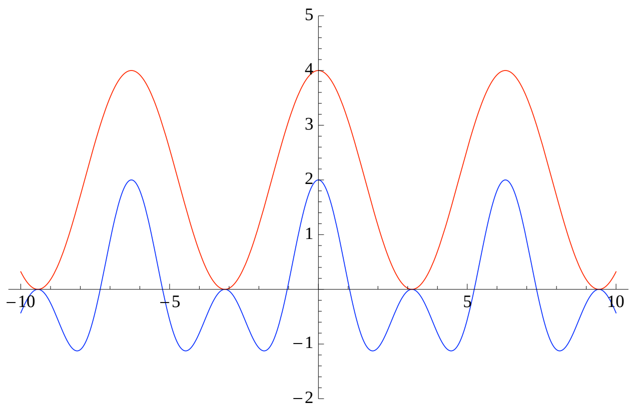


- $\psi_5(x) = \delta(x-1) + \delta(x-2)$.

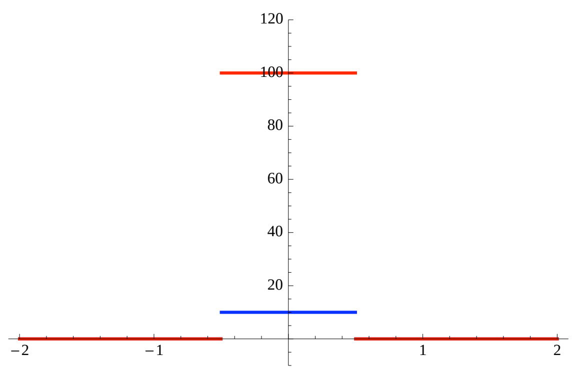


- Real part of the wavefunction: $\mathrm{Re}(\psi_6(x)) = \mathrm{Re}(e^{i2x} + e^{ix}) = \mathrm{Re}(\cos 2x + i \sin 2x + \cos x + i \sin x) = \cos 2x + \cos x$.

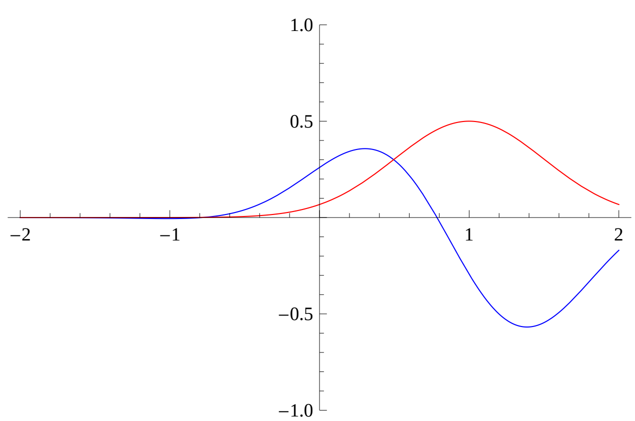
  Probability distribution: $|\psi_4(x)|^2 = \psi_4^*(x)\psi_4(x) = (e^{-i2x} + e^{-ix})(e^{i2x} + e^{ix}) = 2 + e^{ix} + e^{-ix} = 2 + 2\cos x$



- $\psi_7(x) = N$ if $-\frac{a}{2} \leq x \leq \frac{a}{2}$. The x-axis is plotted in units of a, while the y-axis is plotted in units of $0.1N$.



- $\psi_8(x) = Ne^{-\frac{(x-x_0)^2}{a^2}} e^{ik_0 x}$. The x-axis is plotted in units of a, while the y-axis is plotted in units of $0.1N$, $x_0 = 1$ and $k_0 = 2$.



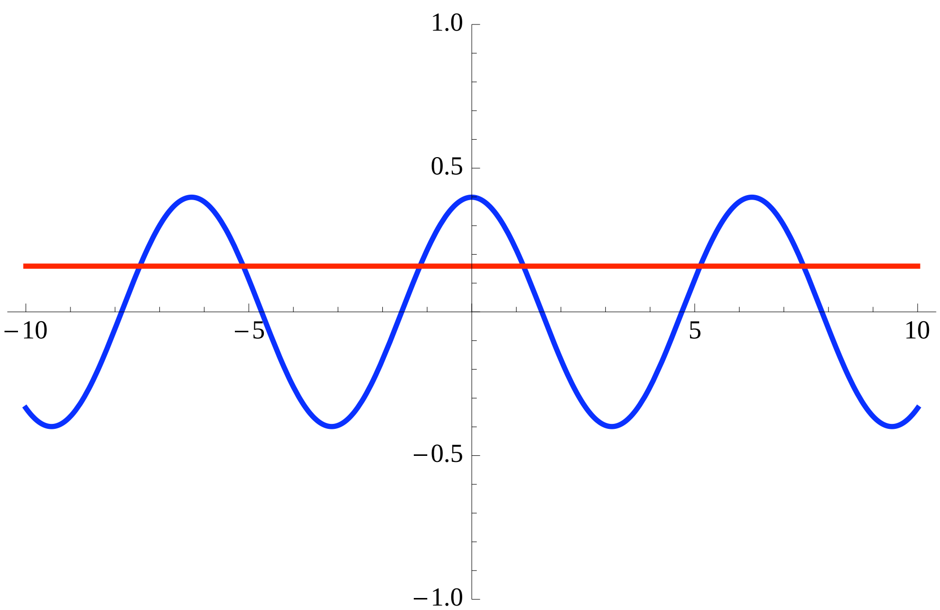
(b) **(4 points)** In each case we must find $\tilde{\psi}(k) \equiv \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \psi(x) e^{-ikx} dx$.

- $\tilde{\psi}_1(k) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \delta(x-1) e^{-ikx} dx = \frac{1}{\sqrt{2\pi}} e^{-ik}$.
- $\tilde{\psi}_2(k) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \delta(x-2) e^{-ikx} dx = \frac{1}{\sqrt{2\pi}} e^{-i2k}$.
- $\tilde{\psi}_3(k) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{ix} e^{-ikx} dx = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{i(1-k)x} dx = \sqrt{2\pi}\delta(k-1)$. (See Problem 6 if you're not sure where the last equality came from).
- $\tilde{\psi}_4(k) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{i2x} e^{-ikx} dx = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{i(2-k)x} dx = \sqrt{2\pi}\delta(k-2)$.
- $\tilde{\psi}_5(k) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} (\delta(x-1) + \delta(x-2)) e^{-ikx} dx = \frac{1}{\sqrt{2\pi}} (e^{-ik} + e^{-i2k})$.
- $\tilde{\psi}_6(k) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} (e^{ix} + e^{i2x}) e^{-ikx} dx = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{i(1-k)x} + e^{i(2-k)x} dx$ $= \sqrt{2\pi}(\delta(k-1) + \delta(k-2))$.
- $\tilde{\psi}_7(k) = \frac{N}{\sqrt{2\pi}} \int_{-a/2}^{a/2} e^{-ikx} dx = N\sqrt{\frac{2}{\pi}} \frac{\sin(ka/2)}{k}$.
- $\tilde{\psi}_8(k) = \frac{N}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-(x-x_0)^2/a^2} e^{ik_0 x} e^{-ikx} dx = \frac{Na}{\sqrt{2}} e^{-i(k-k_0)x_0} e^{-(k-k_0)^2 a^2/4}$.

Note that the operation of taking a Fourier transform is a linear one, so another way to find ψ_5 and ψ_6 would've been to compute $\psi_1 + \psi_2$ and $\psi_3 + \psi_4$ respectively.

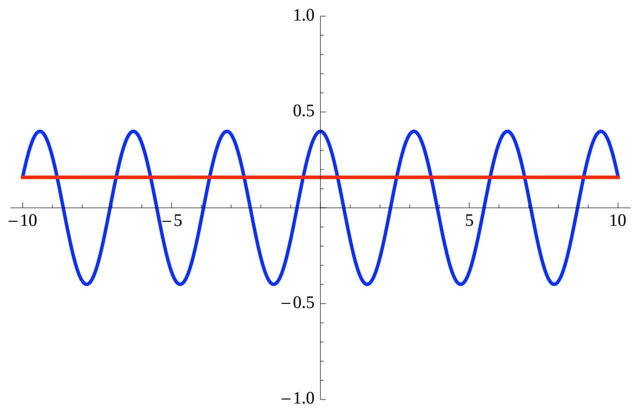
(c) **(4 points)** Again, the real part of the wavefunction is in blue and the probability distribution is in red:

- $\tilde{\psi}_1(k) = \frac{1}{\sqrt{2\pi}} e^{-ik}$, so the real part is $\frac{1}{\sqrt{2\pi}} \cos k$.
  Probability distribution: $|\tilde{\psi}_1(k)|^2 = \tilde{\psi}_1^*(k)\tilde{\psi}_1(k) = \frac{1}{2\pi} e^{ik} e^{-ik} = \frac{1}{2\pi}$.



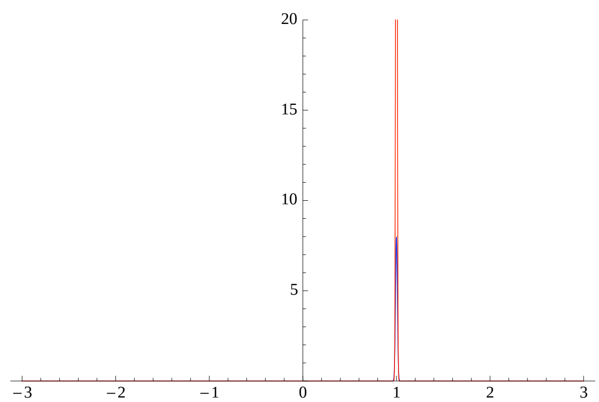
At large k, $\mathbb{P}(k) \sim \frac{1}{2\pi}$. From the formula obtained in Problem 4, this makes $\langle p^2 \rangle$ divergent.

- $\tilde{\psi}_2(k) = \frac{1}{\sqrt{2\pi}} e^{-i2k}$, so the real part is $\frac{1}{\sqrt{2\pi}} \cos 2k$.
  Probability distribution: $|\tilde{\psi}_2(k)|^2 = \tilde{\psi}_2^*(k)\tilde{\psi}_2(k) = \frac{1}{2\pi} e^{i2k} e^{-i2k} = \frac{1}{2\pi}$.



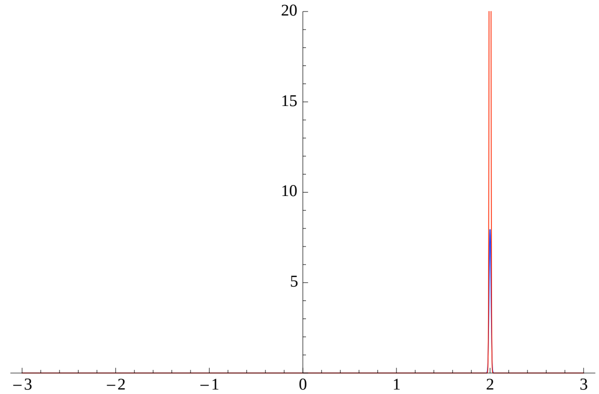
  Also in this case, since $\mathbb{P}(k)$ tends to a constant at large k, the integral for $\langle p^2 \rangle$ is divergent.

- $\tilde{\psi}_3(k) = \sqrt{2\pi}\delta(k-1)$.



  Here $\mathbb{P}(k) = 0$ at $k = 1$. In this case the uncertainty on p is zero.

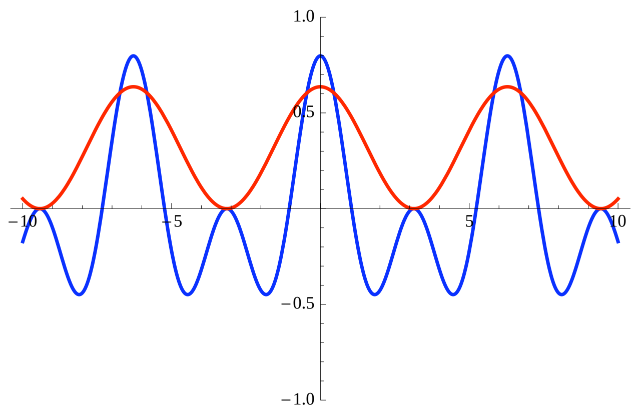
- $\tilde{\psi}_4(k) = \sqrt{2\pi}\delta(k-2)$.



As in the previous case, $\mathbb{P}(k) = 0$ at $k \neq 2$, and the uncertainty on p is zero.

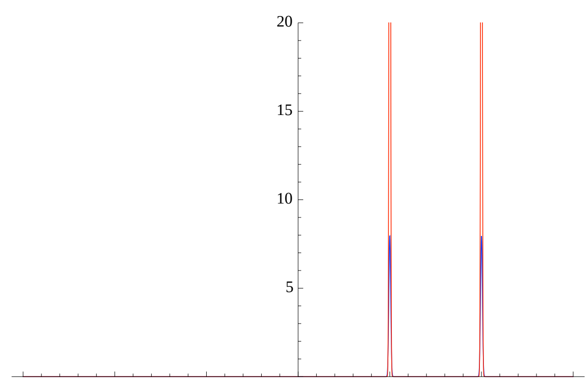
- $\tilde{\psi}_5(k) = \frac{1}{\sqrt{2\pi}}(e^{-ik} + e^{-i2k})$, so the real part is $\frac{1}{\sqrt{2\pi}}(\cos k + \cos 2k)$.

Probability distribution: $|\tilde{\psi}_5(k)|^2 = \tilde{\psi}_5^*(k)\tilde{\psi}_5(k) = \frac{1}{2\pi}(e^{-i2k} + e^{-ik})(e^{i2k} + e^{ik}) = \frac{1}{2\pi}(2 + e^{ik} + e^{-ik}) = \frac{1}{\pi}(1 + \cos k)$.



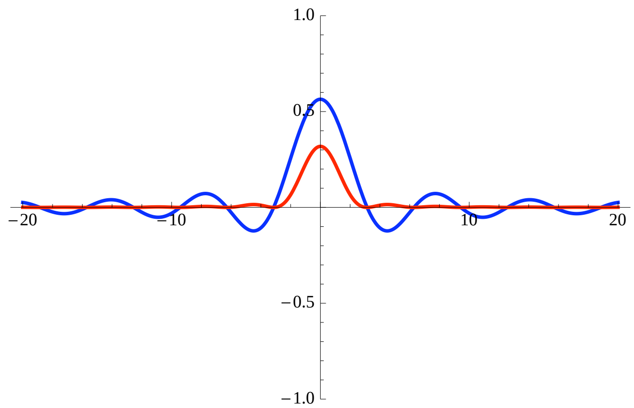
The asymptotic behavior of $\mathbb{P}(k)$ makes $\langle p^2 \rangle$ divergent.

- $\tilde{\psi}_6(k) = \sqrt{2\pi}(\delta(k-1) + \delta(k-2))$.



Since $\mathbb{P}(k) = 0$ for $k \neq 1, 2$, $\langle p^2 \rangle$ is finite, although, contrary to ψ_3, ψ_4, the uncertainty on p is nonzero.

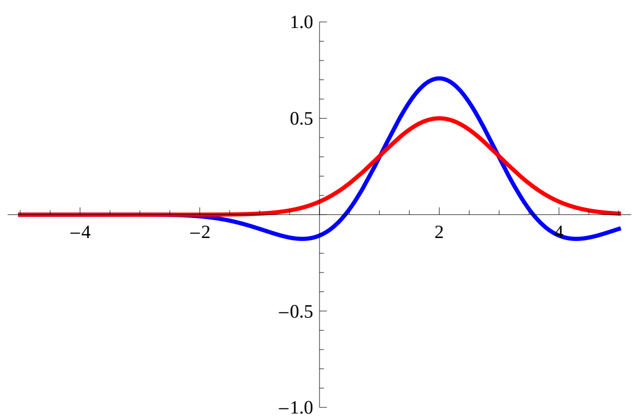
- $\tilde{\psi}_7(k) = N\sqrt{\frac{2}{\pi}}\frac{\sin(ka/2)}{k}$.



$\mathbb{P}(k) = N\frac{2}{\pi}\frac{\sin^2(ka/2)}{k^2}$, so, for large k, $\mathbb{P}(k)$ goes essentially like $1/k^2$, which tends to zero too slowly to obtain a finite value of $\langle p^2 \rangle$:

$$\langle p^2 \rangle = \int_{-\infty}^{\infty} dk (\hbar k)^2 \mathbb{P}(k) \sim \int_{-\infty}^{\infty} dk \sin^2(ka/2) = \infty.$$

- $\tilde{\psi}_8(k) = \frac{Na}{\sqrt{2}} e^{-i(k-k_0)x_0} e^{-(k-k_0)^2 a^2/4}$. The x-axis is in units of $ka/2$, while the y-axis is in units of aN, $x_0 = 1$ and $k_0 = 2$.



In this case, $\mathbb{P}(k) \sim e^{-2(k-k_0)^2 a^2/4}$, and it tends to zero fast enough to get a finite value of $\langle p^2 \rangle$.

(d) **(6 points)** In general, one should look for maxima in $|\psi(x)|^2$ to decide where it will most likely be found. The expectation value $\langle x \rangle$ can be thought of as an "average" position. With momentum, one repeats the same procedure but using $|\tilde{\psi}(k)|^2$ instead. The narrower the peaks of these probability distribution functions, the more confident we can be that the particle's position and momenta are measured to be near their "most likely values".

- Particle #1:
  - Will certainly be found at $x = 1$, because $|\psi(x)|^2 = 0$ everywhere else.
  - Momentum equally likely to take on any value, because $|\tilde{\psi}(k)|^2$ is constant.
- Particle #2:
  - Will certainly be found at $x = 2$, because $|\psi(x)|^2 = 0$ everywhere else.
  - Momentum equally likely to take on any value, because $|\tilde{\psi}(k)|^2$ is constant.
- Particle #3:
  - Particle equally likely to be found anywhere, because $|\psi(x)|^2$ is constant.
  - Momentum will certainly be $\hbar k = \hbar$, because $|\tilde{\psi}(k)|^2 = 0$ for all $k \neq 1$.
- Particle #4:
  - Particle equally likely to be found anywhere, because $|\psi(x)|^2$ is constant.
  - Momentum will certainly be $\hbar k = 2\hbar$, because $|\tilde{\psi}(k)|^2 = 0$ for all $k \neq 2$.
- Particle #5:
  - Particle will either be at $x = 1$ or $x = 2$, with equal probability
  - Momentum most likely $p = 2n\pi\hbar$, where n is any integer.
- Particle #6:
  - Particle most likely to be found at $x = 2n\pi$, where n is any integer.
  - Momentum will be either $\hbar k = \hbar$ or $\hbar k = 2\hbar$ with equal probability.
- Particle #7:
  - Particle equally likely to be found anywhere between $x = -a/2$ and $x = a/2$.
  - Momentum will most likely be zero, but there is considerable width to this probability maximum, so measuring a value significantly different from zero would not be surprising.
- Particle #8:
  - Particle most likely to be found at $x = x_0$, but peak is wide, so a spread in measured position would not be surprising.
  - Momentum will most likely be at $\hbar k = \hbar k_0$, but peak is wide, so a spread in measured momentum would not be surprising.

(e) **(2 points)** We require the total probability of finding a particle *somewhere* to be 1, so properly normalized wavefunctions must satisfy

$$1 = \int_{-\infty}^{\infty} |\psi(x)|^2 \, dx \tag{55}$$

For $\psi_8(x)$, this means

$$1 = \int_{-\infty}^{\infty} |\psi(x)|^2 \, dx = N^2 \int_{-\infty}^{\infty} e^{-2(x-x_0)^2/a^2} \, dx = N^2 a \sqrt{\frac{\pi}{2}} \quad \Rightarrow \quad N = \left(\frac{2}{\pi a^2}\right)^{\frac{1}{4}}. \tag{56}$$

(f) **(4 points)**

As noted above using symmetry, $\langle x\rangle = x_0$ and $\langle p\rangle = \hbar k_0$ for $\psi_8(x)$. If one wishes to be formal, one can compute the integrals explicitly. For example, $\langle x\rangle$ for $\psi_8(x)$ is given by

$$\langle x\rangle = \int_{-\infty}^{\infty} \psi_8^*(x) x \psi_8(x)\, dx = \sqrt{\frac{2}{\pi a^2}} \int_{-\infty}^{\infty} e^{-2(x-x_0)^2/a^2} x dx = x_0. \tag{57}$$

$$\langle p\rangle = \int_{-\infty}^{\infty} \tilde{\psi}_8^*(k) \hbar k \tilde{\psi}_8(k)\, dk = \frac{a}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-(k-k_0)^2 a^2/2} \hbar k dk = \hbar k_0. \tag{58}$$

The uncertainties are given by $\Delta x \equiv \sqrt{\langle x^2\rangle - \langle x\rangle^2}$ and $\Delta p \equiv \sqrt{\langle p^2\rangle - \langle p\rangle^2}$, so our next step is to find $\langle x^2\rangle$ and $\langle p^2\rangle$.

For ψ_8 we have

$$\langle x^2\rangle = \int_{-\infty}^{\infty} \psi_8^*(x) x^2 \psi_8(x)\, dx = \sqrt{\frac{2}{\pi a^2}} \int_{-\infty}^{\infty} x^2 e^{-\frac{2(x-x_0)^2}{a^2}}\, dx = \frac{a^2}{4} + x_0^2 \tag{59a}$$

$$\begin{aligned}\langle p^2\rangle &= \int_{-\infty}^{\infty} \tilde{\psi}_8^*(k) \hbar^2 k^2 \tilde{\psi}_8(k)\, dk = \\ &= \frac{a}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \hbar^2 k^2 e^{-\frac{(k-k_0)^2 a^2}{2}}\, dk = \frac{\hbar^2}{a^2} + (\hbar k_0)^2,\end{aligned} \tag{59b}$$

so $\Delta x = a/2$ and $\Delta p = \hbar/a$. This means $\Delta x \cdot \Delta p = \hbar/2$, so the Gaussian wavefunction saturates the Uncertainty Principle *i.e.* it is a minimum uncertainty wavepacket.

(g) **(2 points)** The position space wavefunction $\psi_8(x)$ gets narrower and taller as $a \to 0$. We thus expect Δx to tend to zero as this happens, which is confirmed by what we found in part (f), where $\Delta x \propto a$.

Conversely, the momentum space wavefunctions $\tilde{\psi}(k)$ get broader and flatter, and Δp tends to infinity. This is again confirmed by the fact that with ψ_8, we have $\Delta p \propto 1/a$.

Problem 7. (15 points) Why the Wavefunction should be Continuous

(a) **(7 points)** The process here is exactly the same as what we did for parts (e) and (f) of the Problem 6. First we find N:

$$1 = \int_{-\infty}^{\infty} |\psi_7(x)|^2 \, dx = N^2 \int_{-a/2}^{a/2} dx = N^2 a \quad \Rightarrow \quad N = \frac{1}{\sqrt{a}}. \tag{60}$$

Now we find $\langle x^2 \rangle$ because we need it to find Δx:

$$\langle x^2 \rangle \;=\; \int_{-\infty}^{\infty} \psi_7^*(x) x^2 \psi_7(x) \, dx = \frac{1}{a} \int_{-a/2}^{a/2} x^2 dx = \frac{a^2}{12}. \tag{61a}$$

Since $\langle x \rangle = 0$ by symmetry (work it out explicitly if you're skeptical!), we have $\Delta x \equiv \sqrt{\langle x^2 \rangle - \langle x \rangle^2} = \sqrt{\langle x^2 \rangle}$, and

$$\Delta x = \frac{a}{2\sqrt{3}}. \tag{62}$$

The spread Δx is thus smaller than the value we found for ψ_8 in the previous problem.

(b) **(8 points)** We found in Problem 6 that

$$\tilde{\psi}_7(k) = \frac{N}{\sqrt{2\pi}} \int_{-a/2}^{a/2} e^{-ikx} dx = N \sqrt{\frac{2}{\pi}} \frac{\sin(ka/2)}{k} = \sqrt{\frac{2}{\pi a}} \frac{\sin(ka/2)}{k}. \tag{63}$$

By symmetry, $\langle p \rangle = 0$. As for $\langle p^2 \rangle$, we have

$$\langle p^2 \rangle = \int_{-\infty}^{\infty} \tilde{\psi}_7^*(k) \hbar^2 k^2 \tilde{\psi}_7(k) \, dk = \int_{-\infty}^{\infty} \frac{2\hbar^2}{\pi a} \sin^2\left(\frac{ka}{2}\right) dk = \infty. \tag{64}$$

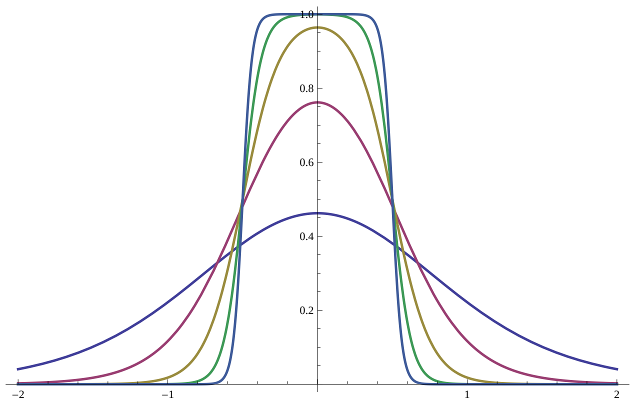
To have a finite $\langle p^2 \rangle$, we need the integral in its corresponding formula to be convergent. Since the integrand is a nonnegative function, we must require that, for large k,

$$k^2 |\tilde{\psi}(k)|^2 \leq \frac{C^2}{k^{1+\epsilon}},$$

for some constants C and ϵ. This implies, for large k,

$$|\tilde{\psi}(k)| \leq \frac{C}{k^{\frac{3+\epsilon}{2}}}.$$

(a) The following picture shows the plots of $\psi_{7b}(x)$ for $b = 1, 1/2, 1/4, 1/8, 1/16$, and $a = N = 1$. Note how, as $b \to 0$, the wavefunction approaches $\psi_7(x)$.
From the picture we note that, for $b << a$, the area between the graph of $\psi_{7b}(x)/N$



and the x-axis is approximately given by a rectangle centered at the origin of height 1 and width a. The same goes for the area between $(\psi_{7b}(x)/N)^2$ and the $x - axis$, and thus $N \sim 1/\sqrt{a}$.

(b) As can be found with Mathematica, or consulting a table of Fourier transforms, we know that the Fourier transform of $\tanh(x)$ is, up to a divergent constant

$$i\sqrt{\frac{\pi}{2}}\text{csch}\left(\frac{k\pi}{2}\right). \tag{65}$$

Because the Fourier transform is linear, and since we will subtract two tanh's, the constants we are neglecting will cancel, so we can ignore them. From (65) we now obtain the Fourier transform of $\tanh(Ax + B)$. Suppose $\tilde{f}(k)$ is the Fourier transform of $f(x)$. The Fourier transform of $f(Ax)$ is then

$$\frac{1}{\sqrt{2\pi}}\int dx e^{-ikx} f(Ax) = \frac{1}{\sqrt{2\pi}|A|}\int dy e^{-ik\frac{y}{A}} f(y) =$$

$$= \frac{1}{|A|}\frac{1}{\sqrt{2\pi}}\int dy e^{-i\frac{k}{A}y} f(y) = \frac{1}{|A|}\tilde{f}\left(\frac{k}{A}\right),$$

where we made the change of variable $y = Ax$. The Fourier transform of $f(x + B)$ is

$$\frac{1}{\sqrt{2\pi}}\int dx e^{-ikx} f(x + B) = \frac{1}{\sqrt{2\pi}}\int dy e^{-ik(y-B)} f(y) =$$

$$= e^{ikB}\frac{1}{\sqrt{2\pi}}\int dy e^{-iky}f(y) = e^{ikB}\tilde{f}(k),$$

where we made the change of variable $y = x + B$. Putting together these results, we find that the Fourier transform of $\tanh\left(\frac{x+\frac{a}{2}}{b}\right)$ is

$$ie^{\frac{ika}{2}}|b|\sqrt{\frac{\pi}{2}}\text{csch}\left(\frac{kb\pi}{2}\right).$$

Using the linearity of the Fourier transform, we finally obtain

$$\tilde{\psi}_{7b}(k) = \frac{N}{2}\left[ie^{\frac{ika}{2}}|b|\sqrt{\frac{\pi}{2}}\text{csch}\left(\frac{kb\pi}{2}\right) - ie^{\frac{-ika}{2}}|b|\sqrt{\frac{\pi}{2}}\text{csch}\left(\frac{kb\pi}{2}\right)\right] =$$

$$= N\sqrt{\frac{\pi}{2}}\frac{|b|\sin\ \frac{ka}{2}}{\sinh\ \frac{kb\pi}{2}},$$

and since $b > 0$, this is the result we were expected to find.

(c) We have that, for large k,

$$|\tilde{\psi}_{7b}(k)|^2 = N^2\frac{\pi}{2}\frac{b^2\sin^2(\frac{ka}{2})}{e^{|k|\pi b}},$$

which means that $|\tilde{\psi}_{7b}(k)|^2$ dies exponentially at infinity, whereas we saw that $|\tilde{\psi}_7(k)|^2$ is precisely equal to zero out of a bounded domain. From the considerations made in Problem 7 (b), we know that $\langle\hat{p}^2\rangle$ is certainly finite.

(d) By symmetry, $\langle\hat{p}\rangle = 0$. Thus $\Delta p^2 = \langle\hat{p}^2\rangle$, and we have

$$\Delta p^2 \propto \int dk k^2 \frac{1}{a}\frac{b^2\sin^2(\frac{ka}{2})}{\sinh^2(\frac{\pi kb}{2})} =$$

$$= \int\frac{dy}{b}\left(\frac{y}{b}\right)^2\frac{1}{a}\frac{b^2\sin^2(\frac{ya}{2b})}{\sinh^2(\frac{\pi y}{2})} \approx \int\frac{dy}{b}\left(\frac{y}{b}\right)^2\frac{1}{a}\frac{b^2/2}{\sinh^2(\frac{\pi y}{2})} \propto \frac{1}{ab},$$

where we made the change of variable $y = kb$. Thus we conclude that

$$\Delta p \propto \frac{1}{\sqrt{ab}}.$$

8.04 Quantum Physics I
Spring 2013