YEN-JIE LEE: So welcome back, everybody, to 8.03. Happy to see you again. So here is the current status of the 8.03. So right now, we have finished the discussion of coupled oscillator. And then we go to infinite number of coupled oscillators.

And we found that there's a wave equation coming out of it. And that means, in short, waves are really a group effort. So many, many objects are working really together, so that they create a wave phenomena. And you can also see, there's a close connection between vibration of a single object and the formation of the wave structure.

So what we are going to do today is to give you a short review of what we have done last time. Then we actually will continue to our understanding of wave equation today. So what we have learned last time, we have learned how to solve infinite system with space translation symmetry.

And also, we learned how to use it to solve finite systems by imposing-- or adding boundary conditions. That would limit the infinite number of normal modes to finite number of normal modes based on-- I mean, it's actually closely related to how many objects you have in the system.

And also, we went ahead and go to a continuum limit. And we found out, there is a surprising result coming out of this. And this is actually the wave equation.

So what do we mean by going to continuous limit? So the limit we are talking about is that, before when we discussed this closed system of infinite number of objects holding together by strings, there is a length scale, which is the separation between objects, which is called a in my notation. And to make it continuous, we are taking a limit such that the a, which is the separation between objects, is so much smaller than the wavelength.

Basically, the wavelength is actually the sinusoidal shape you see when I perturb a system. And what I assuming is that the distance between objects are so much smaller than the wavelengths. So that's actually what I called continuous limit.

And that is actually true for most of the example, which we see in the previous lectures. For example, I was holding a giant spring. And I oscillate that. And all the little components, or, say, all the little mass on that spring, the space between all those little mass on the spring are so much smaller than the length scale we are talking about, which is at the order of 1 meter. So that actually is a sensible limit, which describes the physics we are interested.

When we go to a continuous limit, we find that something really interesting happens. So M minus 1 K matrix originally is infinite times infinite dimension matrix. It becomes the operator, which is actually minus T over rho L partial square partial x squared. And also--

OK, I changed the notation here. It was aj, and I changed it to psi, because what we are going to use later on, when we describe wave functions, et cetera, especially in 8.04, we usually use psi. And the psi j, which were discrete and evaluated in the individual discrete position in the x direction, it's becoming a continuous function, which is psi x and is also a function t.

Therefore, from this exercise, we found out we see wave equation, actually after we go to a continuous limit. And for more information, you can also take a look at the textbook in the relevant page.

So what are we going to do today? So today, what we are going to do is to understand the wave equation, the structure, and what does that mean, and also what are the normal modes coming out of this wave equation. And the next time, in later lectures, we will also discuss another special kind of motion, which is progressing wave solutions.

So let's immediately gets started by looking at a concrete example and also to derive the normal modes from this wave equation. Before we do that, let's take a look at this wave equation.

This wave equation is actually equivalent to infinite number of equations of motion, if you think about it. Why is that? That is because each x-- each partition x you put in will produce a equation of motion. So basically, originally, when we were doing a discrete case, those are labeled by c. c is actually telling you which mass I'm talking about. Now, it's actually replaced by x.

And what we are actually doing is to solve infinite number equation of motion in one go. And that is actually the wave equation. So the first question we ask is, what is actually normal modes based on this infinitely long continuous system described by wave equations? So let's get started immediately.

So basically, we can first assume what is actually the functional form for normal modes. So what we can actually do is we can assume that psi x, t is actually equal to A, is actually a function of x, times B, is actually a function of time. So what I am doing is actually have a meaning actually.

So A of x actually give you a description of the functional form-- the shape of the normal mode as a function of x. So that's actually giving you the shape as a function of x-axis. And B, which is a function of t, is actually giving you information how individual component goes up and down or move as a function of time. So that actually control the time evolution. And we were using this wave equation to describe a continuous system, like, for example, a string with tension t.

So what we could do is the following. So we are interested in the solution of the wave equation, which is shown there. So what we could do is that, OK, let's first assume this functional form, assuming that every component is actually following the same time-depend evolution. And then we can actually plug in this functional form to the equation of motion and see what we will get.

So if we plug this in into the wave equation, so what we are going to get is, if you look at the left hand side, it's actually a partial derivation with respect to time. Therefore, what we are going to get is A of x times partial square of B, which is a function of t, partial t squared. The right hand side of the equation, which is actually equal to vp squared, is a partial derivative with respect to x. Therefore, you have B is a function of t only times partial square A of x partial t squared.

So actually, we can just for convenience-- oh, sorry, that's supposed to be partial x. Thank you very much-- partial x squared. So just for convenience, we can actually divide the whole equation by A times B times the vp squared. We can issue divide the whole equation by A times B, for example, and the vp squared.

If we do this, then basically I'm moving this part to the left hand side. So I get 1 over vp square B of t partial square Bt partial t square. And the right hand side, because I also divide AB vp square, therefore, I get 1 over A of x partial square A of x partial x square. So far so good.

And basically, what I'm doing is just plug in the functional form, which I assume here and then divide everything by AB times vp square. And what I immediately find is that left hand side is a function which only depends on t. Left hand side only depends on t.

And right hand side is a function which on the depends on x. So in short, I have in this situation f of t, which is left hand side, is equal to g of x. You will see this over and over again in later lectures related to physics. This is actually the so-called separation of variables.

So basically, you are facing a situation f of t equal to g of x. If you think about this situation, that's actually really, really helpful, because, OK, now what I can do is I can stay at a specific x. For example, I choose this point. Then I let the time go forward. Of course, I cannot stop time, but if it goes forward, then left hand side equivalent, if it's changing, you will change, because I change t.

If the left hand side equation is changing, then that's actually not going to-- this equation is not going to work, because I am not changing x. I am fixing myself at a specific location, and a lot of time go on. Then if left hand side is changing, then this equation cannot work. You see?

Therefore, what is the consequence of this equation? That means, left hand side, f of t, must be a constant. Therefore, no matter what I do, if I change time, it's not going to change anything. I can put in whatever time, like 1 billion years after this lecture or now, it doesn't matter. It's a constant. So this must be a constant.

I can do the same trick. I froze the time. I fix the time, and then I compare this point to that point, or, say, something billion billions of light years away from this class room. I am changing the x, but I'm not changing the t.

The same argument also holds-- if this function is changing as a function of x, then I am screwed, because--

[LAUGHTER]

--it doesn't work, right? I mean, this-- Therefore, it has to be a constant also as well. Constant also equal to a constant, that's really lovely, right? [LAUGHS] That means, I can say this is equal to that is equal to a constant. As usual, I call this constant really, really strange fancy name-- minus Km square, which you will not like it. But later, you would like it.

[LAUGHTER]

Very good. So we make tremendous amount of progress. Originally, we saw that we are in trouble. It's A times Bt-- Ax times Bt, sounds really horrible. Now, actually, you see that this equation is really simple to solve.

So let's actually take a look at the solution to the f function and the g function. So the first thing, if I take the left hand side, which is a time-dependence part, I can copy there-- this is actually 1-- copy the equivalent here-- is 1 over vp square B of t partial square Bt partial t square. And this is equal to a fancy name of this constant-- minus Km square.

And of course, I can multiply everything by vp squared B. And I get partial square B partial t square. And this is equal too minus vp square Km square B. Wait a second. We have solved this equation infinite number of times in this lecture.

You remember the solution? What is the solution? Anybody can help me?

AUDIENCE: [INAUDIBLE].

YEN-JIE LEE: Anybody? It's sine or cosine function, right? This is harmonic oscillation. It's almost like equation of motion of a spring-mass system, right? I hope that you are already bored. And that means I very successful.

[LAUGHTER]

B of t will be equal to-- [LAUGHS]-- B of m sine omega m t plus beta m, where omega m is actually equal to vp times Km. I define omega m equal to vp time Km. So surprisingly, the solution of B is really simple. It's actually Bm sine omega m t plus beta m.

So what does that mean? That means the overall motion or overall time-dependent evolution of the system is like harmonic motion. If you do get individual component in this system. So that's really nice.

So let's take a look at the right hand side. The right hand side what we have is 1 over A A of x partial square A x partial x squared. And now, this equal to minus Km squared. I don't want to go over this again. This is actually the same thing as number one.

The only thing which is different is that now the partial derivative is actually the x. It's partial square A partial x square. Therefore, I can immediately write down the solution. A of t-- oh, sorry-- A of x will be equal to Cm sine Km x plus alpha m. Any questions so far?

So until now, you accept the fact that f of t and the g of x has to be the same constant. And I call it minus Km square. And I didn't actually tell you what Km I'm choosing. In reality, according to this result, Km can the anything, can be any number, as long as it's a constant.

Therefore, I would like to write the corresponding psi, which is the wave function, is actually labeled by m. m is it just label which K I was using, nothing fancy. It's just a label. Psi of m is a function of x and t.

And that will be equal to Bm times Cm sine omega mt plus beta m sine Km x plus alpha m. Bm and Cm are just arbitrary constant. Therefore, I can merge them. And I will call it just Am.

And, of course, don't forget we have this condition. Omega m is actually equal to vp times Km. So this is actually defined here.

So here, since this is actually a second order differential equation, you have unknown factor, which is beta m. You have also Bm is a unknown. And the right hand side, you also have a Cm, which is unknown, and alpha m, which is unknown.

When we combine them, I replace Bm times Cm by Am. Therefore, what we have is that Am is actually some kind of amplitude, which can be determined by initial conditions, which I will talk about that later. Beta m is the unknown coming from the left hand side derivation.

Alpha m is also unknown, which is actually coming from the right side derivation related to the shape of the normal mode of the system. And finally, there's one additional unknown coefficient, which is Km-- it's kind of arbitrary now-- control actually the wave number, or say the wavelength of the shape of the normal mode.

So when you see this, doesn't this surprise you? May not surprise you any more, because we have solved infinite number of coupled oscillator. And you have learned that, OK, the normal modes have a shape of sine function. It's like a sine function-- before it was like a sine as a Ka. And the Ka is performing x.

So what we are actually getting here is that doesn't surprise you since this system also satisfy space translational symmetry. Therefore, the functional form of the shape of the normal mode is also a sine function. So that's actually pretty satisfactory and also come out as what we would expect based on what we actually have learned.

So let's actually take a look at the structure of this function. So basically, as I mentioned before, everything is oscillating at the frequency omega m with a phase beta m. So that satisfy the condition of normal mode, what is the condition. All the components in the system are oscillating at the same frequency and the same phase. Indeed, yes, that's correct.

Also as a function of time, it's actually going up and down harmonically as we already discussed. And the relative amplitude, as I said, is a sine function. And of course, I already demonstrated this before, that you can see that in this is system, you can see a sine function when I start to drive it.

So what I am doing is actually to convert the kinetic energy from my hand to energy stored as potential or kinetic energy in this string-rod system and in this bell wave machine. So you can see that beautifully those are sine function. And, of course, if I do a higher frequency one, you can see the hand oscillation frequency changed.

And that is actually controlled by this equation, this dispersion relation. And this dispersion relation is actually relating two physical quantity. One is actually the wave number.

Of course, if you are more familiar with wavelength, it's actually 2 pi over Km. Lambda m is actually the wavelength of the shape of the normal mode. And the oscillation frequency is actually controlled by this dispersion relation, this function.

And you can say that, Professor Lee, I have been so tired of this demo. I've seen this 1,000 times, right? And basically, you are showing me that, OK, you can actually oscillate this system and excite this system so that it's oscillating at some natural frequency the system like, right? It's actually some kind of resonance behavior.

So I can actually excite-- I can-- no. I can randomly shake this system. And then it's going to be a linear combination of all the excited normal modes, right? We have seen this many, many times.

What I am going to show you is that there's another machine here, which is actually demonstrating the resonance of some wave. So here is actually so-called the Rijke tube. So the structure is like this. So basically, you have a metal tube, which is red thing there. And inside the tube-- you cannot see it now-- but inside the tube, there's a wire mesh, which the air can flow freely up and down in this mesh.

And what I'm going to do now is to heat up the mesh and see what is going to happen. I will just heat it up by like six second and see what is going to happen. So now, I'm going to do this very carefully.

[RESONATING SOUND]

Can you hear that? [LAUGHS] OK, very good. So listen, what is happening? So you hear a mono frequency sound generated from what? Generated from the heat I gave to the wire mesh.

So what is actually happening? So when I heat up the mesh, what is going to happen is that the air around this screen is going to be heated up. Therefore, because the air is heated up, it goes in the upward direction. And also, the volume of the air is expanding, because of the increased temperature. And that actually goes through this system.

And why it does is really like what I'm doing to the bell lab wave machine, is actually tyring to oscillate-- or excite any possible normal modes, which this system actually like. So you can see that, after a while, once the pressure and also the air inside the tube get then self-organized, then you hear a very loud sound. So that means there are energy flowing from the tube to your ear. And that is actually coming from what? Coming from the heat I put into the system. So it's actually a heat sound wave conversion.

So I hope you enjoyed this demo. And we will take a five-minute break to take questions before we move on. And of course, you are welcome to come here and to play with the demo if you want. [LAUGHS]

So welcome back from the break. So what we are going to do next is to understand how to determine all those unknown coefficients. So you get to see here, there are Am, which is the amplitude. There are beta m, which is basically the phase. There are Km, which is actually the wave number, and alpha m, which is the phase for the shape.

So what I'm going to show you is that Am and the beta m, these two quantity will be determined by initial conditions. Well, Km and alpha m, as you may guess, those can be determined by boundary condition for the Km and alpha m. So why don't we just immediately get started with a concrete example.

So let's take a look at this situation. So this equation and those all the possible Km are allowed when we talk about infinitely long system. So far, we have not imposed any boundary conditions. And what I'm going to do now is to show you a example boundary condition and see how we can actually fix Km and alpha m.

So suppose we are interested in this system. So I have a wall in the left hand side. And I have a string with length L. And it's actually connected to massless ring. And this ring can actually move up and down a long rod in the right hand side.

And I also assume that this string have a constant tension T. And also the density is rho L. So basically, it's a mass per unit length. So that's the system which I am interested.

And, of course, I need to define my coordinate system as usual. I define horizontal direction to be x direction. And I define the vertical direction to be y direction. And I define y equal to 0 is the equilibrium portion of the string.

When psi is equal to 0, that means this string is actually at rest. And not moving-- is actually in the equilibrium position, it's not displaced at all with respect to y equal to 0. And I can also define that x equal to 0 is the position of the left hand side wall.

So this is actually the physical situation. And I would like to actually find out what are the boundary conditions. So what are the boundary condition? So from what we actually discussed last time, left hand side, since this string is actually fixed on the wall, I nailed it there, it cannot move.

Therefore, what is actually the first boundary condition? Why is actually the first boundary condition? Anybody can tell me? Which describes the situation, the physical situation on your left hand side?

AUDIENCE: y0 is 0.

YEN-JIE LEE: y0 is 0. Very good. So when x is equal to 0, y0 is 0. So on my note, I was using a different notation. So I would just use psi. So psi 0 is equal to 0.

Apparently, there's another boundary of this system. The other boundary condition is actually happening at x equal to L. What is actually the boundary condition? Can somebody help me? Yes.

AUDIENCE: Is it the derivative of psi is 0?

YEN-JIE LEE: The derivative of the psi is equal to 0. So we will explain to everybody why is that the case.

The answer proposed is that partial psi partial x L t is equal to 0. And this is 0, t, because that has to be true no matter when I actually invented this boundary condition.

So what is actually giving us this strange boundary condition? So suppose I focus on the force diagram on this ring. So this ring is actually connected to a string with string tension T. Also, there's another force which is actually trying to balance the string tension, which is a normal force-- normal force coming from the rod, which is actually trying to stop the ring from moving in the horizontal direction. So there's normal force then.

And we also know that this ring is actually massless. So m is equal to 0. Suppose that this partial psi partial x, the slope is not 0. If slope is not 0, that means the string may be pulling this ring to some direction. What is going to happen?

So if this happens, it is actually clear that the normal force cannot balance the string for us. Everybody get it? What will happen? If this happened, then this massless ring will suffer from infinitely large acceleration. Because F is equal to ma. And m is 0, so a goes to infinity.

So that means this ring will- peeew- disappear, go to the edge of the universe. Did that happen? No, it didn't happen. Therefore, this condition must be satisfied. You see?

So the slope of the string cannot be nonzero. Otherwise, some crisis will happen. Very good. So we have the two conditions.

And the second thing, which we are going to demonstrate you, is that, OK, I promise you that boundary condition can fix these two constants. So therefore, we are going to demonstrate that. So let's use the first condition we have in the right hand side board.

And basically, from 1, you can actually get psi m 0, t. I am plugging in this condition-- plugging in this solution to boundary condition number 1. And basically, what I am going to get is this is equal to Am sine alpha m sine omega m t plus beta m. And this is actually equal to 0.

So you only have a alpha m here because I am setting x to be equal to 0. I'm setting x to be equal to 0. Therefore, you already have that functional form. So now, we are facing a choice.

So you can set Am to be equal to 0 is arbitrary number. But if you set m equal to 0, everything is 0. And it's not fun, it's not moving. Therefore, I don't want to set Am to be equal to 0.

And you can say, huh, maybe this is equal to 0-- sine omega m t plus beta m is equal to 0. But this is really a sine function. And this condition has to be satisfied no matter at which time you are revisiting this boundary condition. At all times, this boundary condition has to be satisfied. Therefore, this cannot be equal to 0.

Therefore, I conclude that this is the 0. So what does that mean? That means I can choose alpha m is equal to 0. So that's actually given by the first boundary condition.

So let's actually take a look at the second boundary condition-- partial psi partial x evaluated at x equal to L and any time t is equal to 0. So now, I can plug in, again, the solution in the middle board partial psi m L, t, partial x. And that will be equal to Am Km sine omega m t plus beta m cosine Km x. And this is equal to 0.

So I am taking a partial derivative partial psi partial x. Therefore, the sine become cosine. The sine Km x plus alpha m becoming cosine. And also, I know already from the first boundary condition, alpha m is equal to 0. Therefore, I get cosine Km x here.

And this is evaluated at x equal to L. So that means this thing must be equal to 0 based on the second boundary condition. Of course, we can have a losing argument-- Am should not be equal to 0. Otherwise, you will be equal to 0 all the time, the whole wave function is 0.

And this is actually changing as a function of time, the same argument, because this boundary condition has to be satisfied at all times. From the beginning of the Universe to the end of the Universe, this condition has to be satisfied. Therefore, these cannot be equal to 0.

And what is actually left over is cosine Km x evaluated at L equal to 0. So cosine Km L is equal to 0. So that means you cannot arbitrarily choose Km anymore.

Before we introduced boundary conditions, we were saying, ah, Km is actually some arbitrary constant. And now, it's not arbitrary any more. It has to satisfy this condition.

What does that mean? This means that Km has to be equal to 2m minus 1 divided by 2L times pi. You can actually check this. And this small m is equal to 1, 2, 3, et cetera, et cetera. And then you can see that there are many, many different solutions.

So you can see that, as I mentioned before, the boundary conditions determine Km and alpha m. So you can see that the first condition at x equal to 0 determine alpha m. The second boundary condition also help us to determine what are the possible Km value. And that is actually listed here. Any questions? No?

So in order to help you to visualize what we have learned from here, I can now choose m is equal to 1. So you can see that I carefully choose my notation from the beginning. So therefore, m is now the index of the normal mode I am referring to.

So now, if I choose m equal to 1, then I can actually evaluate what would be the resulting K. So K1, according to last formula 2 minus 1 is giving you 1. Therefore, you'll get pi over 2L.

And, of course, you can also calculate based on the wave number what will be the wavelength. So the wavelength lambda 1 will be equal to 2 pi over K1. Wine And that will give you 4L.

Don't forget, once you actually already decide K, the omega is also determined, because omega, which is the angular frequency of this normal mode, is determined by that dispersion relation omega m equal to vp times Km. So therefore, I can now calculate omega 1. That will be equal to vp times K1. And that will you square root of T over rho L pi over 2L. So this is actually coming from the last lecture, the formula of vp.

So that means, if you fix the shape of you are normal mode, then the angular frequency is also fixed, according to this dispersion relation. So of course, I can now visualize this situation. And basically, I can plot this system as a function of time-- as a function of x, not as a function of time.

So when this system reach the maxima amplitude, it would look like this. And this is actually amplitude A1. Because I am talking about the first normal mode labeled by m equal to 1, and there is an unknown amplitude A1. And that is actually showing here.

Of course, I can go ahead and calculate if m is equal to 2, what is going to happen? If I increase the m, what is going to happen is that K is also increased. So K is increased. Then that means the wavelength is decreased.

I have calculated the K for you. And that is equal to 3 pi over 2L. And those are the lambda 2 will be equal to 4L over 3. You can actually double check this at home.

And of course, I can now demonstrate what would be the resulting shape of the massless mode. It would look like this. And this is essentially telling you the amplitude A2. You can also do m equal to 3. If you doing that, basically what you get is something like this, et cetera, et cetera. Any questions?

And the motion of this system that is a function of time is that this whole shape, this shape, is multiplied by sine omega m t plus beta m. So the whole shape is going to scale up and down harmonically. And so if you focus on one of the point here, it's going to be going up and down harmonically.

Very important, there's no back and forth movement. Everything is only moving up and down. If you focus on only one of the particle in this string, everything is moving up and down.

Like here, right? So when I create a curve-- when I create some kind of wave, all the components are always moving only up and down, instead of back and forth, because they can't. They can't move back and forth. But you maybe cheated by the shape-- the evolution as a boundary of time, it seems to me that, ah, something is actually moving back and forth. But never-- all the particles are moving up and down-- very important.

Finally, we have shown you the first three normal modes. And what is actually the most general solution? What is a general solution?

Of course, as we had before, general solution would be a linear combination of all the possible normal modes. So now, I would like to write psi x, t, as the general solution is going to be the sum of all the allowed normal mode. In this specific case, it's going to be a summation from 1-- m equal to 1 to infinity Am sine omega m t plus beta m sine Km x plus alpha m.

And in this specific case, it become cosine-- the Km is there-- cosine 2m minus 1-- sorry, it should be sine. It should be sine 2m minus 1 over 2L pi. And the alpha m in this case is equal to 0.

So the upper formula is the most general case. You're summing over the possible m's. And the Km and alpha m can be determined by boundary conditions. And in this specific case, the right hand side expression is reading like sine 2m minus 1 over 2L times pi.

So that's very nice. And then you can see another sets of example here in the slide. So this is another set of normal modes from m equal to 1 to m equal to 6.

And you can see that in this example both ends are fixed, instead of one end is actually attached to a massless ring. If both ends are fixed, then you get these normal modes. And of course, in the later-- in your p set, you will be exercising this kind of normal modes and solve the corresponding Km and alpha m.

And you can see that, if you focus on the upper left corner, you will see that the oscillation frequency is low. Why is that? That is because the wave number Km is small, therefore, wavelength is long.

According to that formula, omega m is proportional to wave number. Therefore, you can see that the oscillation frequency is actually two times slower compared to m equal to 2 case, which is the upper right corner result And you can see that, if you increase m more and more, you get larger and larger K. And therefore, you see that the oscillation frequency is getting larger and larger.

So now, we are actually facing an issue here. Wait a second, so now we have solved the functional form of the normal mode. We have learned how to determine Km and alpha m.

But we are facing a difficulty here, because Am is very difficult to solve, because you have infinite number of terms here. And beta m, how do we solve this? So it's getting really, really difficult.

So what I am going to tell you is that we can actually, again, use the help from the math department. They have actually proposed the solution. They actually say that, huh, this is actually identical problem that we solved in the math department, is just for the decomposition and for the series.

So what is actually for the series? So you can see, from here, there's a triangular shape between 0 and 1. It's a function-- probably is a function of x. And between 0 and 1, it looks like a triangle. And if you do for the decomposition, it can be decomposed as small k sine function plus the second normal mode and plus a second massless mode.

And you can see that, if you increase the number of terms included in this Fourier series, then you will see that the shape is actually getting closer and closer to the triangular shape. In order to help you with the visualization, here is actually what I prepared. So this actually extracted from essentially a real example, which I really used a computer to calculate. And I tracked the contribution from m equal to 1.

This means that the first term in this summation-- infinite number of term summation-- the first term looks like this. And if you include the first and second and third term, it becomes something like a plateau. And then if you increase 1 to 5, it's evolving as a function of m, becoming more and more-- hm, strange shape. And that is actually including the summation from first term to 11 terms and, finally, 11 to 19.

Huh, what this is-- what is actually the function I put in? It's actually a MIT function!

[LAUGHTER]

I put in a MIT function into this again. And you can see that-- wow, 1 to 59 term. I need to use 59 terms to describe this really wonderful shape. [LAUGHS] So in order to help you with the visualization, listen you can see I prepared a little program, which actually can show you the evolutions as I increase more and more m terms.

Let's take a look. You can see that, originally it looks-- doesn't look-- oh, you cannot see anything. Wait a second. What is going on? Let me see if I can-- I hope I don't screw this up. Sorry. I need to restart.

So let's get started. So you can see that from the first few terms, it doesn't look anything. But very soon, when you have 20 terms added to each other, it looks really pretty much like a MIT dome. And you can see that this program is really trying really hard to describe the sharp edge in the left-hand side and the right-hand side.

You can see that those kind of really infinitely sharp edge will need infinite number of terms, so that if your m is really huge, then the K, the wave number, is going to infinity. Then you can actually produce infinitely sharp edge in this function. And that is actually, you can see from this demonstration the program is really struggling with this really super sharp edge.

So look at the left hand side and right hand side corner, originally the slope is clearly not high enough. And thus, we include higher, higher m value terms. And you can see the description becoming much, much better at the edge but, of course, still are not perfect, because you need infinite number of terms to describe the shape of MIT dome.

Of course, we can also take a look at other example, just testing my eyesight I'm not sure if I-- OK, so I can increase the speed to save on time. So this is actually a square pulse, which you can see from the scope pretty often when you do experiment. And you can see that a square pulse is really difficult to reproduce, as I said before, due to these sharply rising h.

And of course, I can also demonstrate you another example, which is a triangular shape. And you can see that-- ah, still, you can see it works pretty nicely. And the function doesn't like at all the right hand side edge, because of exactly the same reason. OK, very good. So let's come back to the presentation. Can you see-- OK, very good.

So the question is, how do we actually extract Am and beta m? OK, I have done that with a computer program. And what I'm going to do now is to show you a concrete example. And we are going to go through it together to see how we actually can extract Am and beta m.

So suppose I give you an initial condition. It's exactly the same system I am talking about. But now, I prepare this system at t equal to 0 some specific kind of shape. This L/2 is actually the first half of the system, is actually untouched. The first part of the string is actually at the equilibrium position.

And this is actually x equal to L/2. Suddenly, I actually move the string sharply up. And the rest half of the string is actually at the height of h in this case.

And of course, the right hand side edge is x equal to L. And this is actually a snapshot, which I actually took with my camera at t equal to 0. And also, I assume that at t equal to 0, the string-- all the components of the string is at rest.

So based on this information, which I give you, I can now translate this information into mathematics. So that corresponds to two initial conditions. The first one is that, since the string is at rest, that means psi dot x evaluated at t equal to 0 is 0, because the string is at rest. The second initial condition is that psi x 0 is known and is actually shown in this graph.

So from this, we would like to see if we can actually extract information about capital Am and the beta m. So let's immediately get started to see how we can use those initial conditions. So from the first initial condition, a, related to the initial velocity of the string, basically, we can get psi dot x, t. And this will be equal to, let's see, the sum over m equal to 1 to infinity. So basically, I'm taking this equation when I plug in that equation into the first initial condition.

So basically, what I have is Am omega m sine omega m t plus beta m. Oh, this will become cosine-- sorry-- because I'm doing a derivative, psi dot. So this will become cosine. And sine Km x plus alpha m.

And this is actually equal to 0 when psi dot x, t, is actually evaluated at t equal to 0. And this is equal to 0. So if I plug in t equal to 0 to this equation, this becomes 0. And then we know that the shape of the normal mode is some kind of sine function from the previous discussion.

And I am now requiring this thing to be equal to 0. Of course, I cannot make Am omega m equal to 0. That's what we discussed before. And this sine function can be evaluated at any place, any x value. Therefore, this cannot be equal to 0.

Therefore, what is actually the result? The resulting condition is that cosine beta m will be equal to 0. Therefore-- huh, from this initial condition, actually I can conclude that beta m is actually equal to pi/2, for example. So therefore, you can see that very clearly from the first initial condition, the string is not moving at the beginning, I can conclude that beta m is equal to pi over 2.

And just as reminder, alpha m is actually equal to 0 from the previous discussion. And Km is actually equal to 2m minus 1 pi divided by 2L, because I just want to copy here, because somehow the board is covered by another board. So now, I have done with the first initial condition.

And the other initial condition I have is that, OK, I provided you the picture I took at the beginning of the experiment. Therefore, psi x, 0-- at t equal to 0-- is known. So very good. So I have this condition.

But now, I am facing a difficulty, because all those terms-- all the terms, m equal to 1 to infinity-- contribute, as we demonstrated before, to this shape. It's very difficult to actually evaluate Am. So the trick is to make friends from the math department.

So what we could do is that we can use the orthogonality of the sine function to overcome this difficulty. So let me immediately write down what do I mean by orthogonality of the sine function. So if I do a integration from 0 to L on dx sine Km x sine Kn x, if I do this integral, integrating from 0 to L, so what I am going to get is that basically you either get L equal to 2 if m is equal to n, or you get 0, if m is not equal to n.

So basically, I have two sine functions multiplied to each other. And I do integration from 0 to L multiplied by delta x. And this is Km. This is Kn.

If Km and Kn are different, you can actually go ahead and do this exercise. And you will find that, indeed, if they are the same, then you will get L/2. On the other hand, if they are not the same-- the K value are not the same for the first and second sine function-- you are going to get 0.

So that's very good news, because if I do this-- if I do this calculation, I do 2/L integration from 0 to L psi x, 0, sine Km x dx, what is going to happen if I do this integration? Remember, psi m is a linear combination of infinite number of massless modes with different sine Km x. If I multiplied that by sine Km x, this is a very crazy thing to do, because all the other terms will become 0. If the K value of one of the terms is not equal to the dictator's value Km, it's 0. Otherwise, it's L/2, and it is designed here to cancel that factor.

So you can see that this is like a mode picker. I'm picking up a mode with this tool. This is like, this tool, yeah, I'm picking this mode, which is actually matching my Km.

It's a miracle that this become Am. I hope you get this idea, even if probably you haven't heard about with your decomposition before. But essentially, what we are doing is that I'm going to evaluate infinite number of integrals. And you are going to do that in the exam, hopefully easy. [LAUGHS]

What is going to happen is that, if you do this integral, you are going to pick only one mode out of it. And you are going to be able to know the amplitude of that mode. So let's do this immediately in this example. So Am is actually equal to 2/L. Since the amplitude is actually 0 between 0 and L/2. So I can safely just integrate from L/2 to L.

So I do a integration from L/2 to L, because between 0 and L/2, the initial position is 0. So what I'm going to get is h sine Km x dx. And of course, everybody know how to do this integral. It doesn't look that horrible.

And this would become 2/L minus h over Km. And basically, this will become cosine Km evaluated at L minus cosine Km evaluated at L/2, and where this Km, as just a reminder, is basically equal to 2m minus 1 pi divided by 2/L.

So I hope this actually help you to understand the procedure to determining all those unknown coefficients, starting from this equation. Am and the beta m can be determined by initial conditions. As we actually show here, you can use the initial condition of velocity and the initial condition of the shape and the help of a mode picker to pick up the amplitude from that tool function.

And also, you can see that Km and alpha m related to the shape of the normal mode can be determined by boundary conditions-- boundary conditions, how this system is actually connected to the nearby systems. The nearby systems are the rod and the wall. So that is actually the two boundary conditions, which determine the shape of the normal mode.

And finally, very important, as usual, the most general solution is, of course, a linear combination of all of those possible massless modes from m equal to 1 to infinity. And omega m, don't forget, is determined by the dispersion relation, vp times Kl.

Before the end, I would like to mention to you something which you might actually not notice when we were discussing this. So you can see that omega is now proportional to Km. So if you plot omega as a function of k, actually you can see that it's becoming a straight line in this graph, which is very straightforward.

And on the other hand, if you remember what we got last time with discrete system, with length scale between little mass is actually a, you get omega square is equal to 4T over m sine squared ka over 2. So if you plot this omega as a function of k, you will get the black curve. What does this actually tell you? That is actually telling you that, if you prepare a system at a specific normal mode based on the oscillation frequency, you can actually know the internal length scale of individual mass, just in case you didn't notice this interesting fact.

So thank you very much. I hope you enjoyed the lecture. And I will see you next Thursday-- not here in the Walker room-- Walker Memorial. So good luck, everybody. Maybe see some of you in the office hour tomorrow.