8.03 **Fall 2016** Practice Exam **1** Solution

1. (a) When the bottom of the mass falls off the force of gravity on the system decreases by $(1-\alpha)Mg$, so the equilibrium position will move upward a distance $\frac{(1-\alpha)Mg}{k}$. In the coordinate system centered on the initial position of the mass, the new equilibrium position will be

$$y = \frac{(1-\alpha)Mg}{k} \tag{1}$$

(b) As usual, $\omega_0 = \sqrt{\frac{k}{\alpha M}}$. Recall from the first problem set that gravity does not affect the oscillation frequency. Therefore, the period of the oscillations will be

$$\tau = \frac{2\pi}{\omega_0} = 2\pi\sqrt{\frac{\alpha M}{k}} \tag{2}$$

(c) The most general form for the oscillations of this undamped, undriven harmonic oscillator is

$$y(t) = A\cos\left(\omega_0 t + \phi\right) + \frac{(1-\alpha)Mg}{k} \tag{3}$$

where the $\frac{(1-\alpha)Mg}{k}$ term comes from the fact that our system of coordinates is not centered on the equilibrium position of the oscillator. For simplicity we will move to a coordinate system centered on the new equilibrium position for the remainder of the problem. In this new coordinate system the motion of the system will be give by

$$y(t) = A\cos\left(\omega_0 t + \phi\right) \tag{4}$$

(d) Since the system is initially at rest at the equilibrium position of the original (mass M) system, we know that

$$y(0) = -\frac{(1-\alpha)Mg}{k} \tag{5}$$

$$\dot{y}(0) = 0 \tag{6}$$

From (6) and the minus sign in (5) we know that $\phi = \pi$ (note that we could also have said that $\phi = 0$ and included the minus sign in the coefficient). From (5) we can see that the amplitude of oscillations will be $A = \frac{(1-\alpha)Mg}{k}$.

(e) The potential energy stored in the spring will be $PE(y) = \frac{1}{2}ky^2$, so

$$PE(t) = \frac{\left((1-\alpha)Mg\right)^2}{2k}\cos^2\left(\omega_0 t - \pi\right) \tag{7}$$

The kinetic energy of the mass is $KE = \frac{1}{2}\alpha M\dot{y}^2$, and

$$\dot{y}(t) = -\sqrt{\frac{k}{\alpha M}}\frac{(1-\alpha)Mg}{k}\sin\left(\omega_0 t - \pi\right) \tag{8}$$

so

$$KE(t) = \frac{\left((1-\alpha)Mg\right)^2}{2k}\sin^2\left(\omega_0 t - \pi\right) \tag{9}$$

and $E = KE + PE = \frac{((1-\alpha)Mg)^2}{k}$, a constant.

2. (a)

$$A: \quad \ddot{x} = -\frac{k}{m}x - \frac{b}{m}\left(\dot{x} + \omega_d s_0 \sin\left(\omega_d t\right)\right) \tag{10}$$

$$B: \quad \ddot{x} = -\frac{k}{m}\left(x - s_0\cos\left(\omega_d t\right)\right) - \frac{b}{m}\dot{x} \tag{11}$$

So, rearranging to put the driving term on the right,

$$A: \quad \ddot{x} + \Gamma\dot{x} + \omega_0^2 x = -\Gamma\omega_d s_0 \sin\left(\omega_d t\right) \tag{12}$$

$$B: \quad \ddot{x} + \Gamma\dot{x} + \omega_0^2 x = \omega_0^2 s_0 \cos\left(\omega_d t\right) \tag{13}$$

(b) The steady state solution in complex notation will be of the form $z(t) = Ce^{-i\omega_d t}$ for both cases. First, we'll rewrite the equations of motion in complex notation:

$$A: \quad \ddot{z} + \Gamma\dot{z} + \omega_0^2 z = -i\Gamma\omega_d s_0 e^{-i\omega_d t} \tag{14}$$

$$B: \quad \ddot{z} + \Gamma\dot{z} + \omega_0^2 z = \omega_0^2 s_0 e^{-i\omega_d t} \tag{15}$$

Plugging our ansatz into the equations of motion yields

$$A: \quad \left(-\omega_d^2 - i\Gamma\omega_d + \omega_0^2\right)C = -i\Gamma\omega_d s_0 \tag{16}$$

$$B: \quad \left(-\omega_d^2 - i\Gamma\omega_d + \omega_0^2\right)C = \omega_0^2 s_0 \tag{17}$$

Rearranging, we obtain

$$A: \quad C = -\frac{i\Gamma\omega_d s_0}{-\omega_d^2 - i\Gamma\omega_d + \omega_0^2} \tag{18}$$

$$B: \quad C = \frac{\omega_0^2 s_0}{-\omega_d^2 - i\Gamma\omega_d + \omega_0^2} \tag{19}$$

Multiplying the expressions by $\frac{-\omega_d^2 + i\Gamma\omega_d + \omega_0^2}{-\omega_d^2 + i\Gamma\omega_d + \omega_0^2}$ so that the denominators are real, we find

$$A: \quad C = -\frac{\left(i\left(\omega_0^2 - \omega_d^2\right) - \Gamma\omega_d\right)\Gamma\omega_d s_0}{\left(\omega_0^2 - \omega_d^2\right)^2 + \Gamma^2\omega_d^2} \tag{20}$$

$$B: \quad C = \frac{\left(\left(\omega_0^2 - \omega_d^2\right) + i\Gamma\omega_d\right)\omega_0^2 s_0}{\left(\omega_0^2 - \omega_d^2\right)^2 + \Gamma^2\omega_d^2} \tag{21}$$

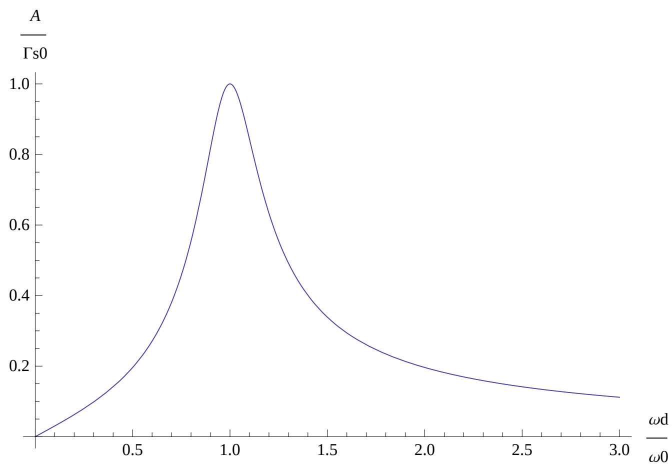
The real amplitude of oscillation is the absolute value of the complex amplitude:

$$A: \quad A = \frac{\Gamma\omega_d s_0}{\sqrt{\left(\omega_0^2 - \omega_d^2\right)^2 + \Gamma^2\omega_d^2}} \tag{22}$$

$$B: \quad A = \frac{\omega_0^2 s_0}{\sqrt{\left(\omega_0^2 - \omega_d^2\right)^2 + \Gamma^2\omega_d^2}} \tag{23}$$

In[16]:=
```
Plot[(0.3 * w) / Sqrt[(1 - w^2)^2 + 0.09 * w^2],
 {w, 0, 3}, PlotRange → All, AxesLabel → {ωd/ω0, A/Γs0}]
```

Out[16]=



In[17]:=
```
Plot[(1) / Sqrt[(1 - w^2)^2 + 0.09 * w^2],
 {w, 0, 3}, PlotRange → All, AxesLabel → {ωd/ω0, A/(ω0^2 s0)}]
```

Out[17]=

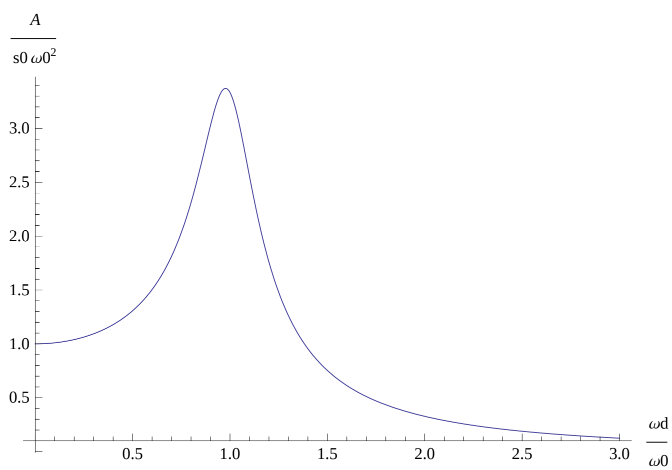


Figure 1: Amplitude as a function of $\frac{\omega_d}{\omega_0}$ for case A (above) and case B (below). $\Gamma = 0.3$

(c) To find the drive frequency which results in the greets amplitude we must take the derivative of the amplitude with respect to ω_d:

$$A: \quad \frac{dA}{d\omega_d} = \frac{\Gamma s_0 \left(\omega_0^4 - \omega_d^4\right)}{\left(\left(\omega_0^2 - \omega_d^2\right)^2 + \Gamma^2 \omega_d^2\right)^{3/2}} \tag{24}$$

$$B: \quad \frac{dA}{d\omega_d} = \frac{-\omega_0^2 s_0 \left(2\omega_d^3 - \left(2\omega_0^2 - \Gamma\right)\omega_d\right)}{\left(\left(\omega_0^2 - \omega_d^2\right)^2 + \Gamma^2 \omega_d^2\right)^{3/2}} \tag{25}$$

Setting these derivatives equal to zero and solving for ω_d yields

$$\begin{aligned} A: \quad \omega_d &= \pm\omega_0 && (26) \\ B: \quad \omega_d &= 0, \pm\sqrt{\omega_0^2 - \frac{\Gamma}{2}} && (27) \end{aligned}$$

Since negative frequencies are unphysical and the zero frequency case is a local minimum, we find that the maximum response is achieved for $\omega_d = \omega_0$ for case A and $\omega_d = \sqrt{\omega_0^2 - \frac{\Gamma}{2}}$ for case B.

(d) In case A, the amplitude of oscillations goes to zero as $\frac{\omega_d}{\omega_0} \to 0$ and it decays to zero as $\frac{1}{\omega_d}$ as $\frac{\omega_d}{\omega_0} \to \infty$. In case B, amplitude goes to $\omega_0^2 s_0$ as $\frac{\omega_d}{\omega_0} \to 0$, and it decays to zero as $\frac{1}{\omega_d^2}$ as $\frac{\omega_d}{\omega_0} \to \infty$, so the amplitude drops off faster as ω_d increases in the second case.

3. (A) We define our coordinates as follows: x_n $(n = 1, 2, 3)$ is the displacement of the n^{th} mass from the left from its equilibrium position, positive x_n to the right. The mass matrix is

$$M = \begin{pmatrix} m & 0 & 0 \\ 0 & 3m & 0 \\ 0 & 0 & m \end{pmatrix} \tag{28}$$

and the k matrix is

$$K = \omega_0^2 \begin{pmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{pmatrix} \tag{29}$$

where $\omega_0^2 = \frac{k}{m}$, so we have

$$\left(\omega_0^2 \begin{pmatrix} 2 & -1 & 0 \\ -\frac{1}{3} & \frac{2}{3} & -\frac{1}{3} \\ 0 & -1 & 2 \end{pmatrix} - \omega^2 I \right) C = 0 \tag{30}$$

(B) Again, x_n $(n = 1, 2, 3)$ is the displacement of the n^{th} mass from the left from its equilibrium position, positive x_n to the right.

$$\left(\begin{pmatrix} \frac{g}{l} + \frac{k}{m} & -\frac{k}{m} & 0 \\ -\frac{k}{m} & \frac{g}{l} + \frac{4k}{m} & -\frac{3k}{m} \\ 0 & -\frac{3k}{m} & \frac{g}{l} + \frac{3k}{m} \end{pmatrix} - \omega^2 I \right) C = 0 \tag{31}$$

(C) We number the masses starting with mass 1 at the top right and moving clockwise around the ring. We define our coordinates as follows: x_n $(n = 1, 2, 3, 4)$ is the displacement of the n^{th} mass from its equilibrium position, positive x_n in the clockwise direction.

$$\left(\omega_0^2 \begin{pmatrix} 4 & -3 & 0 & -1 \\ -3 & 4 & -1 & 0 \\ 0 & -1 & 4 & -3 \\ -1 & 0 & -3 & 4 \end{pmatrix} - \omega^2 I \right) C = 0 \tag{32}$$

(D) Q_n $(n = 1, 2, 3, 4)$ is the charge on the upper plate of the n^{th} capacitor from the left. Notice, however, that these charges are not linearly independent; charge cannot cross from the upper half of the circuit (with the inductors) to the lower half, so the total charge in each half must remain constant. Therefore, we can always solve for the charge on one of the capacitors in terms of the other three.

Next, recall Kirchhoff's junction rule: the sum of currents in a network of conductors meeting at a point is zero. Applying this law at the pint between the left-most capacitor and the left-most inductor we find that the current across the first inductor is simply $\dot{Q}_1 = -I_1$, where I_n is the current across the n^{th} inductor. Applying the rule between the first and second inductors yields $\dot{Q}_2 = I_1 - I_2$. Likewise, $\dot{Q}_3 = I_2 - I_3$. Hence,

$$\begin{aligned} I_1 &= -\dot{Q}_1 && (33) \\ I_2 &= -\dot{Q}_1 - \dot{Q}_2 && (34) \\ I_3 &= -\dot{Q}_1 - \dot{Q}_2 - \dot{Q}_3 && (35) \\ &= \dot{Q}_4 && (36) \end{aligned}$$

To derive the equations of motion, recall that the sum of the potential differences across the circuit elements along any closed loop path through the circuit must be zero. Applying this rule around the three simple loops in the circuit yields

$$\begin{aligned} \frac{1}{LC}(Q_1 - Q_2) - \dot{I}_1 &= 0 && (37) \\ \frac{1}{LC}(Q_2 - Q_3) - \dot{I}_2 &= 0 && (38) \\ \frac{1}{LC}(Q_3 - Q_4) + \dot{I}_3 &= 0 && (39) \end{aligned}$$

If we take the time derivative of each equation and replace all of the derivatives of Q with linear combinations of the currents I_n we obtain the equations of motion in terms of the currents:

$$\begin{aligned} \frac{1}{LC}(2I_1 - I_2) + \ddot{I}_1 &= 0 && (40) \\ \frac{1}{LC}(2I_2 - I_1 - I_3) + \ddot{I}_2 &= 0 && (41) \\ \frac{1}{LC}(2I_3 - I_2) + \ddot{I}_3 &= 0 && (42) \end{aligned}$$

$$\Longrightarrow \left(\frac{1}{LC} \begin{pmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{pmatrix} - \omega^2 I \right) C = 0 \qquad (43)$$

(E) Defining x_n to be the horizontal distance from equilibrium of the n^{th} mass from the top,

$$\begin{aligned} \ddot{x}_1 + \frac{5g}{L}x_1 - \frac{2g}{L}x_2 &= 0 && (44) \\ \ddot{x}_2 + \frac{3g}{L}x_2 - \frac{2g}{L}x_1 - \frac{g}{L}x_3 &= 0 && (45) \\ \ddot{x}_3 + \frac{g}{L}(x_3 - x_2) &= 0 && (46) \end{aligned}$$

so

$$\left(\frac{g}{L} \begin{pmatrix} 5 & -2 & 0 \\ -2 & 3 & -1 \\ 0 & -1 & 1 \end{pmatrix} - \omega^2 I \right) C = 0 \qquad (47)$$

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