Massachusetts Institute of Technology

Physics 8.03

Practice Final Exam 2

Instructions

Please write your solutions in the white booklets. We will not grade anything written on the exam copy. This exam is closed book. No electronic equipment is allowed. All phones, tablets, computers etc. must be switched off.

Formula Sheet Final Exam

Springs and masses:

$$m\frac{d^2}{dt^2}x(t) + b\frac{d}{dt}x(t) + kx(t) = F(t)$$

More general differential equation with harmonic driving force:

$$\frac{d^2}{dt^2}x(t) + \Gamma\frac{d}{dt}x(t) + \omega_0^2 x(t) = \frac{F_0}{m}\cos(\omega_d t)$$

Steady state solutions:

$$x_s(t) = A\cos(\omega_d t - \delta)$$

where

$$A = \frac{\frac{F_0}{m}}{\sqrt{(\omega_0^2-\omega_d^2)^2 + \omega_d^2\Gamma^2}}$$

and

$$\tan\delta = \frac{\Gamma\omega_d}{\omega_0^2 - \omega_d^2}$$

General solutions:
For $\Gamma = 0$ (undamped system):

$$x(t) = R\cos(\omega_0 t + \theta) + x_s(t)$$

where R and θ are unknown coefficients.
For $\Gamma < 2\omega_0$ (under damped system):

$$x(t) = Re^{-\frac{\Gamma}{2}t}\cos\left(\sqrt{\omega_0^2 - \frac{\Gamma^2}{4}}\; t + \theta\right) + x_s(t)$$

where R and θ are unknown coefficients.
For $\Gamma = 2\omega_0$ (critically damped system):

$$x(t) = (R_1 + R_2 t)e^{-\frac{\Gamma}{2}t} + x_s(t)$$

where R_1 and R_2 are unknown coefficients.
For $\Gamma > 2\omega_0$ (over damped system):

$$x(t) = R_1 e^{-\left(\frac{\Gamma}{2}+\sqrt{\frac{\Gamma^2}{4}-\omega_0^2}\right)t} + R_2 e^{-\left(\frac{\Gamma}{2}-\sqrt{\frac{\Gamma^2}{4}-\omega_0^2}\right)t} + x_s(t)$$

where R_1 and R_2 are unknown coefficients.

Coupled oscillators

$$F_j = -\sum_{k=1}^{n} K_{jk} x_k$$

Examples for $n = 2$

$$\mathcal{X}(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}$$

$$\mathcal{K} = \begin{bmatrix} K_{11} & K_{12} \\ K_{21} & K_{22} \end{bmatrix}$$

$$\mathcal{M} = \begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix}$$

Matrix equation of motion, matrices $\mathcal{M}$, $\mathcal{K}$, $\mathcal{I}$ are $n \times n$, vectors $\mathcal{X}$, $\mathcal{Z}$ are $n \times 1$.

$$\frac{d^2}{dt^2}\mathcal{X}(t) = -\mathcal{M}^{-1}\mathcal{K}\mathcal{X}(t)$$

$$\mathcal{Z}(t) = \mathcal{A}e^{-i\omega t}$$

$$(\mathcal{M}^{-1}\mathcal{K} - \omega^2\mathcal{I})\mathcal{A} = 0$$

To obtain the frequencies of normal modes solve:

$$det(\mathcal{M}^{-1}\mathcal{K} - \omega^2\mathcal{I}) = 0$$

For $n = 2$

$$det\begin{bmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{bmatrix} = M_{11}M_{22} - M_{12}M_{21}$$

If the system is driven by force one can find the response amplitudes $\mathcal{C}(\omega_d)$

$$\mathcal{F}(t) = \mathcal{F}_0 e^{-i\omega_d t}$$

$$\mathcal{W}(t) = \mathcal{C}(\omega_d)e^{-i\omega_d t}$$

$$\mathcal{C}(\omega_d) = \begin{bmatrix} c_1(\omega_d) \\ c_2(\omega_d) \end{bmatrix}$$

$$(\mathcal{M}^{-1}\mathcal{K} - \omega_d^2\mathcal{I})\mathcal{C}(\omega_d) = \mathcal{F}_0$$

solving the equation above one can find the response amplitudies for the first ($c_1(\omega_d)$) and second ($c_2(\omega_d)$) objects in the system.

Reflection symmetry matrix:

$$\mathcal{S} = \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}$$

Eigenvalues (β) and eigenvectors ($\mathcal{A}$) of this 2×2 $\mathcal{S}$ matrix:

(1) $\beta = -1$, $\mathcal{A} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

(2) $\beta = 1$, $\mathcal{A} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$

1D infinite coupled system which satisfy space translation symmetry:

Given a eigenvalue β, the corresponding eigenvector is

$$A_j = \beta^j A_0$$

where

$$A_j(A_0)$$

is the normal amplitude of jth(0th) object in the system.

Consider an one dimentional system which consists infinite number of masses coupled by springs, β can be written as $\beta = e^{ika}$ where k is the wave number and a is the distance between the masses.

Kirchoff's Laws (be careful about the signs!)

$$\text{Node}: \sum_i I_i = 0 \ \text{Loop}: \sum_i \Delta V_i = 0$$

$$\text{Capacitors}: \Delta V = \frac{Q}{C} \quad \text{Inductors}: \Delta V = -L\frac{dI}{dt} \quad \text{Current}: I = \frac{dQ}{dt}$$

Trigonometric equalities:

$$\sin(a \pm b) = \sin(a)\cos(b) \pm \cos(a)\sin(b)$$

$$\cos(a \pm b) = \cos(a)\cos(b) \mp \sin(a)\sin(b)$$

$$\sin(a) + \sin(b) = 2\sin\left(\frac{a+b}{2}\right)\cos\left(\frac{a-b}{2}\right)$$

$$\sin(a) - \sin(b) = 2\cos\left(\frac{a+b}{2}\right)\sin\left(\frac{a-b}{2}\right)$$

$$\cos(a) + \cos(b) = 2\cos\left(\frac{a+b}{2}\right)\cos\left(\frac{a-b}{2}\right)$$

$$\cos(a) - \cos(b) = -2\sin\left(\frac{a+b}{2}\right)\sin\left(\frac{a-b}{2}\right)$$

$$e^{i\theta} = \cos\theta + i\sin\theta$$

Some useful integrals involving sin and cos:

$$\frac{2}{L}\int_0^L \sin\left(\frac{n\pi x}{L}\right)\sin\left(\frac{m\pi x}{L}\right)dx = \begin{cases} 1, & \text{if } n = m. \\ 0, & \text{otherwise.} \end{cases}$$

$$\frac{2}{L}\int_0^L \cos\left(\frac{n\pi x}{L}\right)\cos\left(\frac{m\pi x}{L}\right)dx = \begin{cases} 1, & \text{if } n = m. \\ 0, & \text{otherwise.} \end{cases}$$

$$\frac{2}{L}\int_0^L \cos\left(\frac{n\pi x}{L}\right)\sin\left(\frac{m\pi x}{L}\right)dx = 0$$

$$\int x\sin(x)dx = \sin(x) - x\cos(x) + C$$

$$\int x\cos(x)dx = \cos(x) + x\sin(x) + C$$

Maxwell Equations in vacuum

$$\frac{\partial E_y}{\partial x} - \frac{\partial E_x}{\partial y} = -\frac{\partial B_z}{\partial t} \ ; \ \frac{\partial E_z}{\partial y} - \frac{\partial E_y}{\partial z} = -\frac{\partial B_x}{\partial t} \ ; \ \frac{\partial E_x}{\partial z} - \frac{\partial E_z}{\partial x} = -\frac{\partial B_y}{\partial t}$$

$$\frac{\partial B_y}{\partial x} - \frac{\partial B_x}{\partial y} = \mu_0\epsilon_0\frac{\partial E_z}{\partial t} \ ; \ \frac{\partial B_z}{\partial y} - \frac{\partial B_y}{\partial z} = \mu_0\epsilon_0\frac{\partial E_x}{\partial t} \ ; \ \frac{\partial B_x}{\partial z} - \frac{\partial B_z}{\partial x} = \mu_0\epsilon_0\frac{\partial E_y}{\partial t}$$

$$\frac{\partial E_x}{\partial x} + \frac{\partial E_y}{\partial y} + \frac{\partial E_z}{\partial z} = 0 \ ; \ \frac{\partial B_x}{\partial x} + \frac{\partial B_y}{\partial y} + \frac{\partial B_z}{\partial z} = 0$$

Lorentz force

$$\vec{F} = q(\vec{E} + \vec{v} \times \vec{B})$$

Wave equation for EM fields in vacuum

$$\frac{\partial^2 E_i}{\partial x^2} + \frac{\partial^2 E_i}{\partial y^2} + \frac{\partial^2 E_i}{\partial z^2} = \frac{1}{c^2}\frac{\partial^2 E_i}{\partial t^2} \text{ where } \ i = x, y, z$$

$$\frac{\partial^2 B_i}{\partial x^2} + \frac{\partial^2 B_i}{\partial y^2} + \frac{\partial^2 B_i}{\partial z^2} = \frac{1}{c^2}\frac{\partial^2 B_i}{\partial t^2} \text{ where } \ i = x, y, z$$

For EM plane waves in vacuum:

$$\vec{B}(\vec{r}, t) = \frac{1}{c}\hat{k} \times \vec{E}(\vec{r}, t)$$

$$\vec{E}(\vec{r}, t) = c\vec{B}(\vec{r}, t) \times \hat{k}$$

Linear energy density in a string with tension T and mass density ρ_L

$$\frac{dK}{dx} = \frac{1}{2}\rho_L\left(\frac{\partial y}{\partial t}\right)^2 \quad \frac{dU}{dx} = \frac{1}{2}T\left(\frac{\partial y}{\partial x}\right)^2$$

EM energy per unit volume and Poynting vector:

$$U_E = \frac{1}{2}\epsilon_0\vec{E}^2 \quad U_B = \frac{1}{2\mu_0}\vec{B}^2 \quad \vec{S} = \frac{1}{\mu_0}\vec{E} \times \vec{B}$$

Transmission and reflection

$$R = \frac{Z_1 - Z_2}{Z_1 + Z_2} \ , \ T = \frac{2Z_1}{Z_1 + Z_2}$$

Phase velocity and impedance:

$$v=\sqrt{\frac{T}{\rho_L}}\ ,\ Z=\sqrt{T\rho_L}\ \ \text{(string)}$$

$$v=\sqrt{\frac{1}{LC}}\ ,\ Z=\sqrt{\frac{L}{C}}\ \ \text{(transmission line)}$$

Snell's law

$$n_1\sin\theta_1=n_2\sin\theta_2$$

Fourier transform

$$f(t)=\int_{-\infty}^{\infty}d\omega C(\omega)e^{-i\omega t}$$

$$C(\omega)=\frac{1}{2\pi}\int_{-\infty}^{\infty}dt f(t)e^{i\omega t}$$

Delta function

$$\frac{1}{2\pi}\int_{-\infty}^{\infty}e^{i(\omega-\omega')t}dt=\delta(\omega-\omega')$$

$$\int_{-\infty}^{\infty}\delta(x)dx=1$$

$$\int_{-\infty}^{\infty}\delta(x-a)f(x)dx=f(a)$$

Electric and magnetic field from an accelerated charge:

$$\vec{E}(\vec{r},t)=-\frac{q\vec{a_\perp}(t-|r|/c)}{4\pi\epsilon_0 rc^2}$$

$$\vec{B}(\vec{r},t)=\frac{\hat{k}\times\vec{E}}{c}$$

Total power emitted by the accelerated charge:

$$P(t)=\frac{q^2a^2(t-r/c)}{6\pi\epsilon_0c^3}$$

Interference of two sources with amplitudes A_1 and A_2 with a relative phase difference δ:

$$<I>\propto(A_1^2+A_2^2+2A_1A_2\cos\delta)$$

Interference of N fields of equal amplitude with phases $\delta_{m+1}-\delta_m=\delta$:

$$< I >=< I_0 > \left[\frac{\sin(N\delta/2)}{\sin(\delta/2)}\right]^2$$

Single slit diffraction where β is the phase difference between rays coming from edges and the center of the slit:

$$< I >=< I_0 > \left[\frac{\sin(\beta)}{\beta}\right]^2$$

Rayleigh's criterion for resolution: Diffraction peak of one image falls on te first minimum of the diffraction pattern of the second image.

Electric field transmission and reflection ratios, magnitude and sign, for radiation incident normally on an interface between lossless dielectrics with indices of refraction n_1 and n_2.

$$\frac{E_t}{E_i} = \frac{2n_1}{n_1 + n_2} \quad \frac{E_r}{E_i} = \frac{n_1 - n_2}{n_1 + n_2}$$

Schrodinger's Equation

$$i\hbar\frac{\partial}{\partial t}\psi(x,t) = \left[-\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2} + V(x,t)\right]\psi(x,t)$$

where V is the potential energy, m is the mass of the particle and ψ is the wave function.

Problem 1 (15 pts)

Answer each short question separately.

1.1. The potential energy of a particle of mass m, constrained to move along the x-axis is given by:

$$U(x) = A(1 - \cos(\alpha x))$$

where A and α are constants, both > 0.

If the particle is displaced from the equilibrium, what will be its period of small amplitude oscillation?

1.2. Consider the following trace of the position of a driven oscillator as a function of time (Figure 1). You may assume that the driving force is a sine wave and the amplitude of the force does not change over subsequent time. Which of the following description is/are true? (Select all that apply).

(a) The driving frequency is larger than the natural resonant frequency of the system

(b) The driving frequency is smaller than the natural resonant frequency of the system

(c) There is no damping

(d) The system is overdamped

(e) The system is critically damped

(f) The system is underdamped

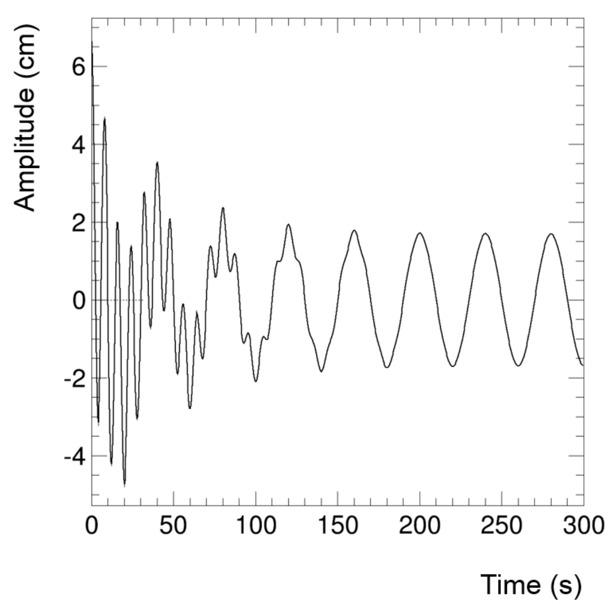

Figure 1: Trace of a driven oscillator

1.3. Electronics used at the Large Hadron Collider use 1 nanosecond square pulses. What is the approximate range of frequencies (bandwidth) required to send such short pulses?

1.4. An electron experiment is shown in Figure 2. The source was heated such that it starts to emit electrons. Which of the following description is/are true? (Select all that apply).

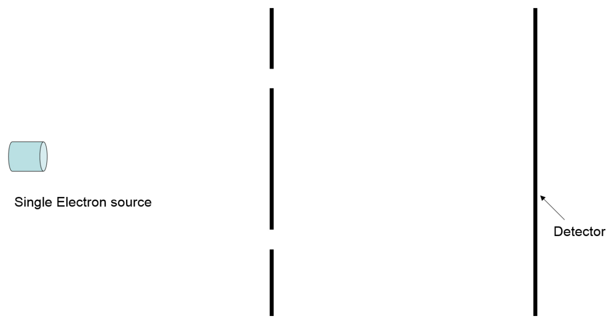

Figure 2: Electron experiment

(a) When the temperature of the source is high such that the rate of electron emission is high, an interference pattern will be recorded by the detector.
(b) When the temperature of the source is high such that the rate of electron emission is high, no interference pattern will be recorded by the detector.
(c) When the temperature of the source is low such the source emits one electron each time, there will be no interference pattern.
(d) When the temperature of the source is low such the source emits one electron each time, there will be interference pattern.

1.5. An elastic membrane is stretched on a rectangular frame as shown in Figure 3. The phase velocity for propagation of waves on this membrane is v. What is the angular frequency of the lowest normal mode that can be excited on the membrane?

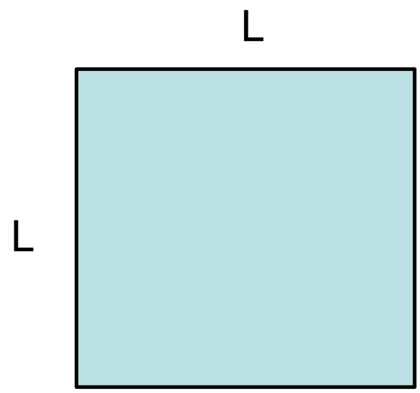

Figure 3: An Elastic Membrane

Problem 2 (15 pts)

Two small massive beads, with equal masses $m_1 = m_2 = m$ are on a taut massless string of length $5L$ (see Figure 4). The tension in the string T is large such that you can ignore the effects of gravity.

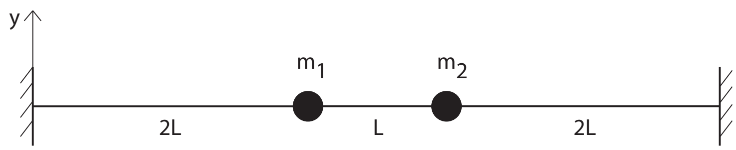

Figure 4: Two beads on a string

a. Write the equations of motion for the two beads for the small amplitude oscillations along y and write matrix $M^{-1}K$ corresponding to this system.

b. Find the shapes and angular frequencies of the normal modes for the system. You can simplify your task by using symmetry arguments. Explain your reasoning.

c. Initially, at $t = 0$, both masses are stationary with m_1 at the equilibrium position and m_2 displaced from the equilibrium by a distance A. Write an expression for the displacement $y_1(t)$ for the mass initially at the equilibrium position.

Problem 3 (20 pts)

Figure 5 represents a gas filled pipe which is open to a gas reservoir at $x = 0$ and closed at $x = L$. The speed of sound in the gas is v. There is a slight pressure disturbance which is established in the gas and then released from rest at $t = 0$. The disturbance is centered at $L/2$, spans a width $L/3$, and has a pressure P_I which is slightly greater than the ambient pressure P_0.

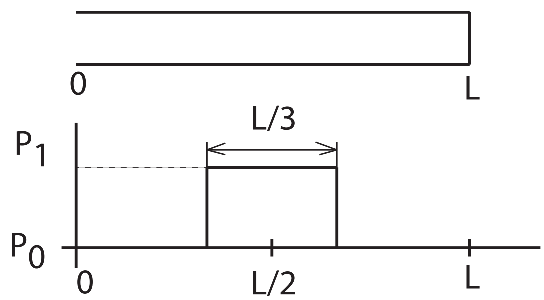

Figure 5: Pressure Wave in a Tube

a. What are the boundary conditions at $x = 0$ and $x = L$?

b. Express the pressure disturbance $P(x, t)$ for $t > 0$ as a sum of normal modes. Give explicit expressions for the spatial and time variations of each normal mode, its wave number, and its angular frequency. Leave the associated amplitudes as parameters to be determined.

c. Calculate the amplitude of the nth normal mode.

d. Draw a sketch (similar to the graph above) of the pressure in the pipe at time $t = 2L/v$. [Hint: This can be done with some careful thought rather than explicitly computing $P(x, t)$.]

Problem 4 (15 pts)

A point charge $+q$ has been moving with constant velocity w along the straight line until the time $t = t_0$. In the SHORT time interval from $t = t_0$ to $t = t_0 + \Delta t$, a force perpendicular to the trajectory changes the direction without changing the magnitude of the velocity. After the time $t = t_0 + \Delta t$ the charge again moves with velocity w along a straight line forming a small angle $\Delta\alpha$ with the initial trajectory as shown in Figure 6. Radiation emitted by the charge is observed from very distant points P_1 andP_2. The two observation points are located in the plane of the trajectory.

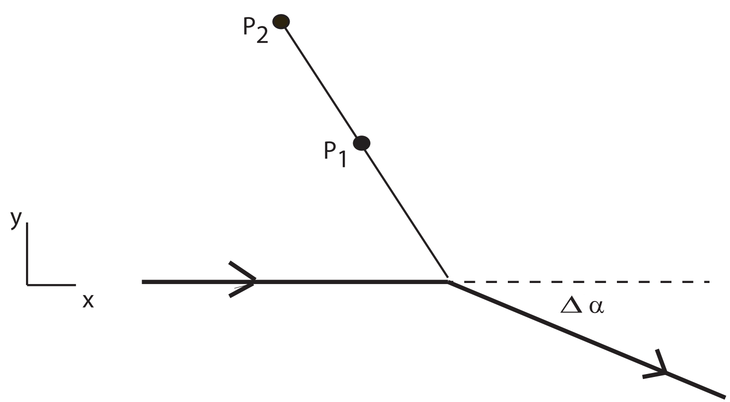

Figure 6: Radiating charge.

a. What is the average acceleration of the point charge in terms of the given quantities?

b. What is the direction of the electric field caused by the acceleration, at the distant point P_1?

c. In what direction is the radiation intensity of the accelerated charge most intense?

d. Where is it least intense?

e. Point P_2 is twice as far from the bend in the trajectory as P_1. By what fraction does the amplitude of the electromagnetic disturbance decrease as the radiation pulse moved from P_1 to P_2?

f. What is the total energy radiated by the charge?

Make careful sketches in answering parts b), c) and d)

Problem 5 (15 pts)

Consider a system of three ideal linear polarizers arranged along an optical bench as shown in the Figure 7. Two outside polarizers have their easy axes perpendicular to each other. Polarizer A transmits only horizontally polarized light while polarizer C transmits only vertically polarized light. Polarizer B has its easy axis at an angle θ to the horizontal x-axis. Assume that the light shining on the polarizer A from the left is unpolarized and its intensity is I_0.

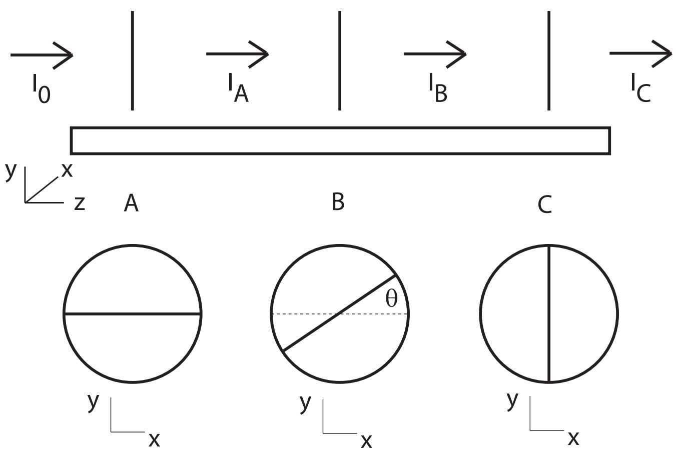

Figure 7: Three Linear Polarizers

a. Find the intensity and polarization of light transmitted through the polarizer A, I_A.

b. Find the intensity and polarization of light transmitted through the polarizer B, I_B, as a function of I_0 and θ and graph it as a function of θ.

c. Find the intensity of light transmitted through the polarizer C, I_C, as a function of I_0 and θ and graph it as a function of θ.

Problem 6 (20 pts)

A monochromatic source of plane waves of wavelength λ illuminates a four slit grating. Figure 8 shows a cross section of the grating; the length of the slits is perpendicular to the page. The screen is very distant from the slits ($d \ll z$).

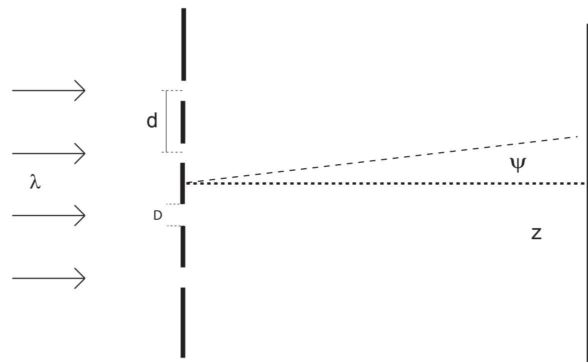

Figure 8: Four Slit Grating

a. Write an expression in terms of d, λ and ψ for the intensity I that will be viewed on the screen.Assume at first that the slits are very narrow compared to their separation ($D \ll d$). Assume that the intensity of light due to one slit is I_0.

b. Make a sketch of the intensity as a function of $\sin\psi$ for the four slit grating. Be sure to specify the locations of the interference principal maxima and minima.

c. Now consider the same grating with the two INNER slits blocked. Write an expression for the intensity observed on the screen and make the sketch of the new intensity versus $\sin\psi$.

d. Compare it to the sketch obtained for the four slits. What are the new locations of the maxima and minima? Which principal maxima are at the same location for the two configurations? How has the magnitude of the principal maxima changed? Assume that the individual light intensities of the open slits are the same for both cases.

e. Consider now the same four slit grating with all slits uncovered, but this time the widths of the individual slits D cannot be ignored. The ratio of the distance between the slit centers to the slit width is now $d/D = 5$. The effect of single slit diffraction will cause some of the principal maxima obtained in a) to disappear (I=0). What is the lowest interference order for which diffraction effects zero out the principal maximum in this fashion?