DR. PETER DOURMASHKIN: I would like to now analyze the motion of a system of particles that has both translational and rotational motion. So I'm going to consider a pulley, and the pulley has radius R. And there is a string wrapped around THE pulley and a block of object 1 that's on a plane, and another block of object 2. And as object 2 falls down, the pulley rotates and object 1 moves to the right. And there's a coefficient of friction between block 1 and the surface.

Now, in order to analyze this problem, I'm going to apply, for the pulley, our torque equals I cm alpha, and for each of the blocks, I'll apply F1 equals m1 a1 and F2 equals m2 a2. But the important thing to realize is that these three quantities, the acceleration of block 1, the acceleration of block 2, and the angular acceleration of the pulley, are constrained because this string is not slipping around the pulley. And so let's begin to analyze this type of problem.

So we'll start with the torque principle. Now, what's crucial in all of these problems is that we're relating two different quantities, vectors on both sides. The physics quantities have definite direction, and our alpha a1 and a2 as vectors are determined by our choice of coordinates. So what I'd like to do is draw a coordinate system, a rotational coordinate system. Now, the way I'll do it is I'll draw an angle theta. And now I have to draw a right-hand move, so my angle theta will look as though it's going into the plane of the figure. And so I write cross n hat right-hand move, and I'm going to just to define that to be k hat.

Now, what that allows me to do, when I write my point s here will be cm. So I'm going to calculate this about the center of mass, and I get I cm alpha. As soon as I draw the coordinate system, then this side becomes the vector alpha z k hat, where alpha z is the z component of the angular acceleration. And technically, the reason this angle is there is because this is the second derivative of that angle. And that's well-defined now.

So the next step is to define the force, to do what we call a torque diagram. So this is my rotational coordinate system. The next step is to construct a torque diagram, and the way we do that is we draw the object. We indicate our rotational coordinate system. I don't have to put the theta anymore.

Now, here's a subtle point. I'm going to draw the rope that is connected to the pulley as part of my system. This is the part where the tension here, I'll call that T2, and over here, this is the tension T1. Now, on the pulley, there is a gravitational force, and there's some pivot force on this pulley. And now what I want to consider is the torque about the Cm. Now, the pivot force, f pivot, and the gravitational force, produce no torque about the pivot, so I'm just going to eliminate those for the moment, and just focus on the torque due to T1 and T2. So I draw my vector Rs T1 and my vector Rs T2.

So this is what a torque diagram consists of. Let's summarize it. It's our system, the relevant forces that are producing torque, vectors from the point we're calculating the torque. Our S is the center of mass. And the vector from where we're calculating the torque to where the force is acting. And now, when we calculate the cross-product of Rs and T, we put these two vectors tail to tail, and notice that this vector is giving us a torque out of the board, our positive direction is into the board, so over here we have minus T1 R. Whereas T2, when we put these two vectors, Rs T2, and we calculate that torque, that torque is into the board, which is our positive direction, and so that's plus T2 R2.

And now in our torque principle, we set these two sides equal, and we have minus T1 R plus T2 R equals Icm alpha z. Now, this is our first equation, but it requires some type of thought. For the first thing, we see that the tension T2 is equal to Icm over R alpha z plus T1. So the tensions on the side are not equal.

Now, when we studied pulleys earlier in the semester, we imposed a condition that the pulley was frictionless, which meant that the rope was sliding along the pulley, and there was no rotation in the pulley, so there was no contribution to alpha. And in that case, T2 would be equal to T2. We also could make a slightly different statement. We could say suppose the mass of the pulley was very, very small, an extremely light pulley, then Icm would be 0, and again, T2 would be equal to T1. So when we were dealing with either massless pulleys or ropes that were slipping frictionlessly along a pulley, the tension on both sides was equal.

Now, something different is happening. We need to apply a greater torque here than T1 because there is rotational inertia. We're causing the pulley to accelerate. So this torque from T2 has to be bigger than the torque from T1, and therefore T2 is bigger than T1, so that is a very important way to apply the torque principle. When T2 is bigger than T1, alpha will be positive, and a positive angular acceleration is giving a rotation in which our angle theta is not only increasing, but its second derivative is positive.

So that's crucial for beginning the analysis of this problem. The next step is to analyze Newton's second law on both objects, M1 and M2. So I'll save our result here, I'll erase what we don't need, and then continue the analysis.

So returning to our analysis of a pulley with two masses and a string that's not slipping around the pulley, I now want to begin analysis of F equals Ma on object 1. So as usual, I draw object 1. I'll choose i hat 1 to point in the direction because I know it's going to go that way so my component of acceleration will be positive. In my force diagrams, I have a normal force, I have gravity. The string is pulling T1, that's the same tension at the end of the string. The tension in the string is not changing. We're assuming it's a massless string. And I have a friction force on object 1, which is kinetic friction.

And now I can write down Newton's second law in the horizontal direction. I could also call j hat 1 up, and my two equations for Newton's second law are T1 minus Fk is M1 A1, and N1 minus M1 g is zero. Now, I also know that the kinetic friction, Fk, is the coefficient of friction mu times N1. So my next equation for F equals M A on object T1 is T1 minus mu N1 equals M1 A1.

Now I have to apply the same analysis to 2. Notice I'm not drawing my force diagram on my sketch. I do a separate force diagram on 2. So here's 2. I have tension T2 and gravity M2g. Now, even though I chose a unit vector up here, this choice of unit vectors is completely independent of how I choose unit vectors for 2. Because object is moving down, I would prefer to choose j hat 2 down. My acceleration for this object will be positive. And then when I apply f equals MA object 2, I get M2g minus T2 equals M2 A2. So that's now my third equation, that M2g minus T2 equals M2 A2.

And now I look at this. System of equations. And what are my unknowns? T1, T2, alpha, A1, A2. Five unknowns. I'm treating properties of the system, the radius mu, Icm, actually the N1, because it's in M1g, I can simplify this equation and replace this with M1g, where I'm already using the other Newton's second law. So I have three equations and five unknowns. I cannot solve this system. But in all of these problems, there's constraint conditions. There's constraints between how the masses are moving and how the angular acceleration pulley is related to the linear acceleration of the masses.

Let's consider mass 1 and 2. They're attached by a string. As mass 2 goes down, mass 1 goes to the right. The string is not stretching, so they're moving at the same rate, so they have the same acceleration. So my first constraint is that A1 equals A2. Now in general, I have to be careful. Plus or minus. Why is it a plus sign and not a minus sign here? It's a plus sign because I've chosen i hat to the right and I've chosen j hat downwards. If I'd chosen them differently, that sign could have varied.

Now, let's focus on the relationship between the angular acceleration of the pulley and M2. Think about the strength. Here we're on a point on the rim. This is a distance R, and the pulley and the string are moving together. So there's a tangential acceleration of the pulley equal to R alpha Z. This is the tangential acceleration of pulley and string. But the same string has a linear acceleration, either A1 or A2. So this has to be equal to A2, this is the linear acceleration of the string. And so that's our last constraint, five, that A1 equals R alpha.

And now I have a system of five equations and five unknowns. And the question is, how can I find the acceleration? So in general, when opposed to a system like that, I want to have some strategy. Let's make a little space to clear for our algebra.

OK, now, I look at this system, and I say to myself, which equation do I want to use as a background? My target is to find A1. A1 is equal to A2. Now, when I look at these equations, T1 depends on A1, T2 depends on A2, which is equal to A1, and alpha is also related to A1. So I can use this equation 1 as my backbone, and substitute in T1, T2, and alpha into that equation. And now let's do that.

So when I solve this equation for T1 equal to M1 A1 plus mu M1g with the minus sign, I get minus M1 A1 plus mu M1g times R, that's my first piece. I solve for T2, which is M2g minus M2 A, so I get M2g minus M2. Now, A2 is equal to A1, so I make my second substitution, multiply it by R, and that's equal to Icm. And now I make my final substitution, that alpha z is equal to A1 over R.

So if I now can collect terms, minus, minus over here, but there's an R there, an R there, I'll divide through by R, and bring my A1 terms over to the other side, and I'm left with minus mu M1g plus M2g equals Icm over R squared that has the dimensions of mass, because moment of inertia, M R squared divided by R squared, plus M1 A1 plus M2 A1. And finally, as a conclusion, I now can solve for the acceleration of my system in terms of all of these quantities. And let's just put it all the way down here at the bottom that A1 equals M2g minus mu M1g over Icm R squared plus M1 plus M2.

Often, in types of problems like these when there's a lot of signers, you might end up with a minus or a minus sign down here, and if you looked at that, that would imply that with the right choice of parameters, this could be zero and that would be an impossible solution. So that's always a sign that there could be something wrong.

The other thing we want to check is, when does it actually accelerate? We have a condition. So we can conclude that if M2g is bigger than Mu M1g then 2 will start to go downwards. If M2g were less than M1g, then the problem would be very different, because two would not go downwards. The friction would not be kinetic, but would be static. And that would vary, depending on how much weight were here. So if you went from 0 to mu M1g, the static friction would depend on how much weight that's there.

So here is a full analysis of rotational and translational motion. Takes a little bit of time and a little bit of care, but we've done it.