PROFESSOR: So in this example, we want to hit an apple hanging from a tree with a projectile. And the main point of the problem is to figure out at what angle to the ground should we aim our projectile when we fire it off in order to hit the apple. And in this example, we're assuming that the apple drops from the tree at the same instant that we fire the projectile.

So in this drawing, let's just put some dimensions in here, we'll assume that the apple starts out at a height, h, above the ground. That the horizontal distance from where the projectile starts to the apple is a distance, d. That the projectile starts at a distance, s, above the ground. And that we fire the projectile off with an initial velocity, v0, at an angle theta, sorry, theta not, with respect to the horizontal.

And let's define our origin to be right here on the ground, directly below where the projectile begins. We need to define our coordinate system, so the i-hat direction will be horizontally to the right, and the j-hat direction will be vertically upwards with an origin at this point. So the question that we have is, what should our angle, theta0 b, in order to hit the apple, which starts falling from the tree at the same instant that we fire our projectile, and we want to hit the apple before it hits the ground.

OK so to work this out, we need to consider the kinematics of two separate objects, our projectile and the apple. So let's begin with the apple. So for our apple, the apple drops vertically straight downwards, so its horizontal position throughout its motion is just unchanged. So the x-coordinate of the apple as a function of time is just d, it's just a constant. It's y-coordinate, it starts out at a height, h, and then it drops due to the acceleration of gravity, and so that's given by minus 1/2 gt-squared. It's a minus because it's falling in the minus j-hat direction.

Now for our projectile, let's call it a bullet. For our bullet, the x-coordinate of the bullet, it starts out at the origin, at x equals zero. And its initial motion, it has an initial velocity, v0, in this direction. The horizontal component of that is v0 times the cosine of theta0. And so it's displacement of time t is v0 t times the cosine of theta 0. And there's no horizontal acceleration, so that completes our x-coordinate as a function of time. For our y-coordinate, we start out at a height, s, above the ground. There is the change in the y-coordinate due to the initial velocity, which has component v0 sine theta0, so that's plus v0 t sine theta 0. And there's also a vertical acceleration due to gravity, which gives us minus 1/2 gt-squared.

OK so that's the kinematics of the apple and the bullet. Now for there to be a hit at time t equals capital-t, the coordinates of the apple and the bullet have to be the same at that collision time. So for a hit at t equals big T, we require that the x-coordinate of the bullet at time big T is the same as the x-coordinate of the apple. And likewise, that the y-coordinate of the bullet is equal to the y-coordinate of the apple.

OK so let's look at each of these in turn. For our x-coordinate, the x-coordinate of the bullet is v0 capital-T times the cosine of theta 0. And that has to be equal to the x coordinate of the apple, which is just d. So notice that I can solve this for the time of the collision, capital-t, and that's just d minus v0 times the cosine of theta 0.

Now for our y-coordinate, the y-coordinate of the bullet is s plus v0 capital-T time is the sine of theta not minus 1/2 gt squared and that's equal to the y-coordinate of the apple, which is h minus 1/2 gt squared.

All right, now I can rearrange that. Notice that I have a minus 1/2 gt squared on both sides, so those cancel out. I can rearrange that to write v0 times capital-T times the sine of theta 0 is equal to h minus s, so that the sine of theta 0 is equal to h minus s over v0 capital T. But remember we solved for capital-T here, so I can substitute that in. So that gives me h minus s over v0 times 1 over t, which is v0 cosine theta 0 over d. And so I can rewrite that as h minus s over d times to cosine of theta 0.

So rewriting that, I have on the left hand side, sine of theta 0 over cosine of theta 0 is equal to h minus s over d. And notice on the left hand side, sine over cosine is just tangent, so this is just the tangent of theta 0. Now think about what that means. If I draw a right triangle where this is theta 0, then this is h minus s and this is d. And so this is the same geometry we have here, if we drew the triangle like this. So theta 0 is just the angle to the location of the apple just before it drops. So what this calculation tells us is that the correct thing to do if we want to hit the apple, if we know the apple is going to drop at the same instant that we fire, then we should aim at the location that the apple is at that instant. We shouldn't try and lead the apple and fire below it or fire above it, we should aimed directly at the apple. And what the kinematics shows us is that both the bullet and the apple fall vertically down at the same rate. And so that will give us a collision.

Now it's worth thinking about two different cases, which I'm not going to solve for you here. The first is, what if the apple didn't drop? Suppose the apple just stayed in the tree and I fired.

If I aimed directly at the apple and the apple didn't drop, then I would miss the apple, because the bullet would drop as it went across this distance, d. Now of course if the apple were big enough, or if the bullet were flying fast enough, if v 0 was fast enough, then it might be that the amount of the bullet dropped wouldn't be bigger than the thickness of the apple, so I might still graze the apple, but I wouldn't hit the apple dead center. So if I knew that the apple wasn't going to drop and I wanted to hit it dead center, I would have to choose some different angle, theta 0. We can sort of see intuitively that that angle would have to be a little bit steeper so that the bullet would drop and then hit the apple. And so the way we would solve that is in our original kinematic equations. For the motion of the apple, x would still be a constant, d, but if the apple wasn't dropping then y of the apple would also be a constant. We wouldn't have the second term, we would just have y equals h. And then I would have to solve for a collision.

The other case to think about, that I'd like you to think about, is what if the apple begins dropping for a short interval before we fired. So that at the time that we fire our projectile, the apple is already dropping. Well that's equivalent to saying that at time equals 0, when I fire, that the apple has some initial velocity, whatever velocity it's picked up by dropping and whatever interval it was dropping before. So now, my kinematics for my apple in the vertical direction, I would have an additional term here, a minus v0 t, where v0 is that initial velocity. And that's a different v0, I should have used a different symbol. It's a different v0 than the v0 of the bullet, but there would be an initial velocity, falling velocity, of the apple that I'd have to consider. So the particular details of what's happening with the apple change the kinematics, and it's worth thinking about how that would change your answer.