8.012 Physics I: Classical Mechanics
Fall 2008

MASSACHUSETTS INSTITUTE OF TECHNOLOGY
Department of Physics

Physics 8.012   Fall 2006

**Exam 2
Thursday, November 2, 2006**

NAME: _____ SOLUTIONS _____

MIT ID number: _____

Instructions:
1. Do all **FIVE (5)** problems. You have **90 minutes**.
2. **SHOW ALL WORK**. Be sure to **circle your final answer**.
3. Read the questions carefully
4. All work must be done in this booklet. Extra blank pages are provided.
5. NO books, notes, calculators or computers are permitted. **A sheet of useful equations is provided on the last page**.

Your Scores

Problem	Maximum	Score	Grader
1	15		
2	15		
3	30		
4	20		
5	20		
Total	100		

Problem 1: Masses and Rod [15 pts]
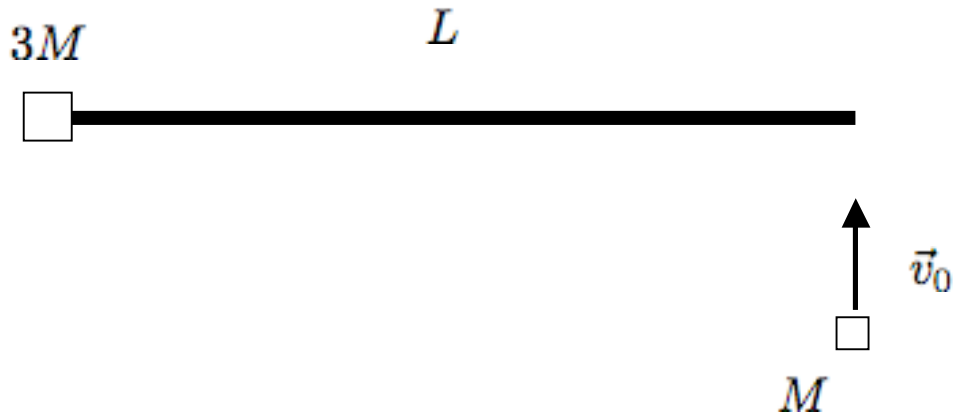
A block with mass 3M connected to a massless rod of length L lies at rest on a fixed frictionless table. A second block of mass M impinges on the system with speed v_0, strikes the opposite end of the rod at a right angle, and sticks. For this problem, assume that the dimensions of the blocks are much smaller than the length of the rod.

(a) [5 pts] What is the final velocity of the center of mass of the block-rod-block system?

(b) [5 pts] What is the final angular velocity of the block-rod-block system about its center of mass?

(d) [3 pts] What is the initial velocity of the more massive block after the collision? Be sure to indicate its direction and speed.

SOLUTIONS

(a) Linear momentum of system is conserved (no external forces), so

$$Mv_0 = 4Mv \Rightarrow \boxed{v = \frac{1}{4}v_0}$$

(b) Angular momentum of system is conserved (no external torques) so

$$L = Ml^2\omega_0 = Mlv_0 = I\omega$$

where l is the distance from the center of mass (COM) to the end of the bar:

$$R_{COM} = \frac{ML}{4M} = \frac{L}{4} \Rightarrow l = \frac{3}{4}L$$

and I is the moment of inertia of two point masses about the COM:

$$I = 3M\left(\frac{L}{4}\right)^2 + M\left(\frac{3L}{4}\right)^2 = \frac{3}{4}ML^2$$

and hence

$$\omega = \frac{Mlv_0}{I} = \frac{(3/4)MLv_0}{(3/4)ML^2} = \boxed{\frac{v_0}{L}}$$

(c) at the position of 3M, total velocity is combination of linear and rotation motion, so

$$v = \frac{1}{4}v_0 - R_{COM}\omega = \frac{1}{4}v_0 - \frac{L}{4}\frac{v_0}{L} = \boxed{0}$$

Problem 2: An Unusual Force Law [15 pts]

Consider the following force law:

$$F(x) = \frac{mc^2}{4}\left(\frac{4a^2}{x^3} - \frac{3a^3}{x^4} - \frac{a}{x^2}\right)$$

for x > 0 , where m is the mass of the object, a is a constant length and c is the speed of light (Note: this force is related to the motion of a mass near a black hole).

(a) [5 pts] Sketch the potential energy of this force law as a function of x. What are the two finite equilibrium points for this force law (i.e., excluding $x = \infty$)? Which of these is stable?

(b) [5 pts] Compute the period of small oscillations about the stable equilibrium point.

(c) [5 pts] An object of mass m, initially at rest at the unstable equilibrium point, moves toward the stable equilibrium point. What is its velocity at the stable equilibrium point?

SOLUTIONS

(a) First calculate potential energy:

$$U(x) = -\int F(x)dx = \frac{mc^2}{4}\left(\frac{2a^2}{x^2} - \frac{a^3}{x^3} - \frac{a}{x}\right) + U_0$$

Now calculate the equilibrium points, where $\dfrac{dU}{dx} = -F(x) = 0$:

$$\left(\frac{4a^2}{x^3} - \frac{3a^3}{x^4} - \frac{a}{x^2}\right) = 0$$

$$\Rightarrow 4ax - 3a^2 - x^2 = 0 \qquad \text{(ignoring } x = \infty \text{ solution)}$$

$$\Rightarrow (x-a)(x-3a) = 0$$

so $\boxed{x = a, 3a}$ are the finite equilibrium points. We can evaluate these points and the limits for x:
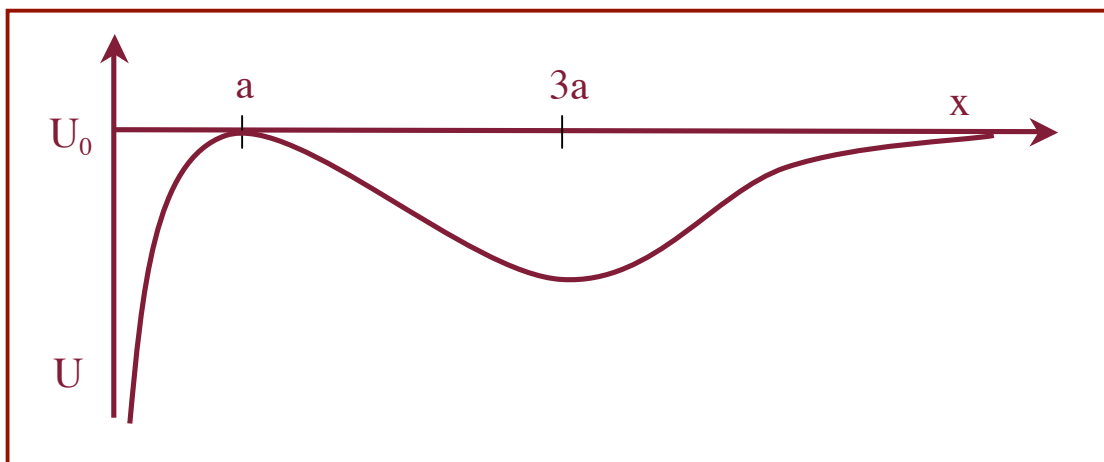
$$U(a) = U_0$$

$$U(3a) = U_0 - \frac{mc^2}{27}$$

$$U(0) \rightarrow -\infty \qquad \text{(note how highest power term is } < 0)$$

$$U(\infty) \rightarrow 0$$

The sketch can then be drawn as below

By inspection, we can see that x=a is unstable and x=3a is stable. We can also determine this from the second derivative of U:

$$\frac{d^2U}{dx^2}(a) = \frac{mc^2}{4}\left(\frac{12a^2}{a^4} - \frac{12a^3}{a^5} - \frac{2a}{a^3}\right) = -\frac{mc^2}{2a^2} < 0$$

$$\frac{d^2U}{dx^2}(3a) = \frac{mc^2}{4}\left(\frac{12a^2}{(3a)^4} - \frac{12a^3}{(3a)^5} - \frac{2a}{(3a)^3}\right) = \frac{mc^2}{162a^2} > 0$$

so $\boxed{\text{x=3a is stable}}$

(b) The effective spring constant about the stability point is

$$k_{eff} = \frac{d^2U}{dx^2}(3a) = \frac{mc^2}{162a^2}$$

The period of small oscillations is therefore:

$$P = \frac{2\pi}{\omega} = 2\pi\sqrt{\frac{m}{k_{eff}}} = 2\pi\sqrt{\frac{162ma^2}{mc^2}} = \boxed{18\sqrt{2}\pi\frac{a}{c}}$$

(c) At x = a, the total mechanical energy is
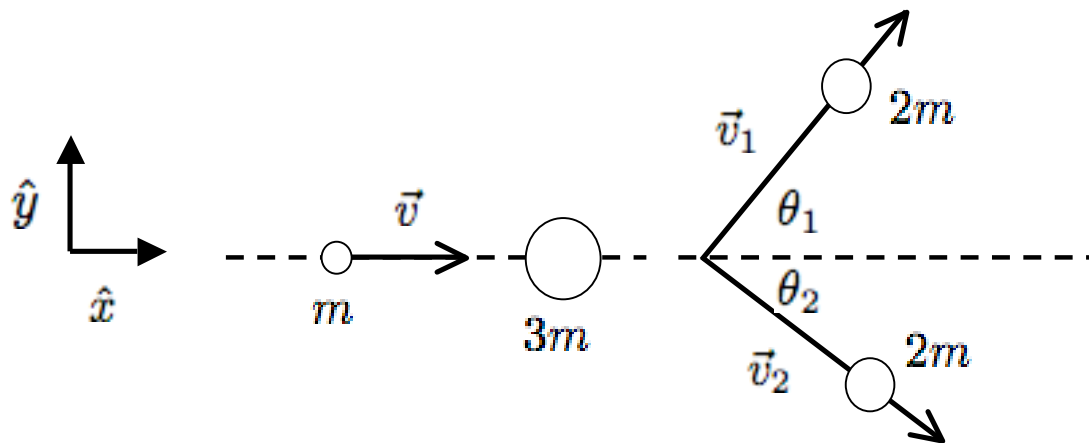
$$E = U(a) = U_0$$

At x = 3a,

$$E = K + U(3a) = K - \frac{mc^2}{27} + U_0$$

As this is a conservative force, total mechanical energy is conserved, so

$$E(3a) - E(a) = 0 = K - \frac{mc^2}{27} + U_0 - U_0$$

$$\Rightarrow K = \frac{mc^2}{27} \Rightarrow \boxed{v = \frac{1}{3}\sqrt{\frac{2}{3}}c}$$

Problem 3: Particle Collision [30 pts]

A particle of mass m and velocity $\vec{v} = v\hat{x}$ collides with a particle of mass 3m at rest in the LAB frame. The ensuing interaction creates two particles with identical masses 2m that scatter off with velocities $\vec{v}_1$ and $\vec{v}_2$ at angles θ_1 and θ_2 from the $\hat{x}$ axis, respectively, in the LAB frame. A mechanical energy Q is lost as a result of the interaction. Assume that there are no external forces at work.

(a) [10 pts] What is the minimum kinetic energy (K_0) of the incident particle required for this interaction to take place?

(b) [10 pts] Assume that the incident particle has initial kinetic energy $K = \alpha^2 K_0$ where $\alpha^2 > 1$. Write an expression relating the scattering angle θ_1 in the LAB frame to the center of mass scattering angle Θ (the deflection angle of the particles in the center of mass frame) that depends only on α.

(c) [10 pts] One of the scattered particles is observed to emerge at an angle of 90° with respect to the $\hat{x}$ axis in the LAB frame. In the limit that the kinetic energy of the incident particle is much larger than the minimum energy (i.e., $\alpha^2 \gg 1$), what is the maximum interaction scattering angle Θ for which this could occur? Note: this can be solved without doing part (b).

SOLUTIONS

(a) The minimum initial K_0^L (in the LAB frame) occurs when the emergent particles have zero velocity in the COM frame, or when they are moving together in the LAB frame. So there are two ways to calculate this. First, move to COM frame:

$$\vec{V}_{COM} = \frac{m\vec{v}}{4m} = \frac{1}{4}v\hat{x}$$

$$\Rightarrow \vec{v}_{1c} = \vec{v}_1 - \vec{V}_{COM} = \frac{3}{4}v\hat{x} \quad \text{and} \quad \vec{v}_{2c} = -\vec{V}_{COM} = -\frac{1}{4}v\hat{x}$$

In both the LAB or COM frame, since there are no external forces:

$$E' = E - Q \Rightarrow K' = K - Q$$

where the prime variables refer to after the collision. Note that this equation only works when we compare energies in the same frame! If the emergent particles have no velocity in COM frame, then

$$K^{C\prime} = 0 \Rightarrow K^C = Q = \frac{1}{2}m\left(\frac{3}{4}v\right)^2 + \frac{1}{2}(3m)\left(\frac{1}{4}v\right)^2 = \frac{3}{8}mv_{min}^2$$

then the minimum kinetic energy of the first particle in the LAB frame is

$$K_0^L = \frac{1}{2}mv_{min}^2 = \boxed{\frac{4}{3}Q}$$

We can also solve this entirely in the LAB frame by noting that if the emergent particles are at rest in the COM frame, they most both have velocity $+\vec{V}_{COM}$ in the LAB frame; hence:

$$K_0^L = \frac{1}{2}mv_{min}^2 = K^{L\prime} + Q$$

$$\frac{1}{2}(2m)\left(\frac{1}{4}v\right)^2 + \frac{1}{2}(2m)\left(\frac{1}{4}v\right)^2 + Q = \frac{1}{8}mv^2 + Q$$

$$\Rightarrow \frac{3}{8}mv^2 = Q \Rightarrow K_0^L = \frac{1}{2}mv^2 = \boxed{\frac{4}{3}Q}$$

(b) Assume:

$$\frac{1}{2}mv^2 = K^L = \alpha^2 K_0^L = \frac{4}{3}\alpha^2 Q \Rightarrow v^2 = \frac{8}{3}\frac{\alpha^2 Q}{m}$$

In the COM frame, the emergent particles scatter with velocities $\vec{v}_{1c}'$ and $\vec{v}_{2c}'$. Since the total linear momentum in the COM frame is 0,

$$\vec{P}_{tot,c} = 0 = 2m\vec{v}_{1c}' + 2m\vec{v}_{2c}' \Rightarrow \vec{v}_{1c}' = -\vec{v}_{2c}'$$

and hence the magnitudes are:

$$|\vec{v}_{1c}'| = |\vec{v}_{2c}'| = v_c'$$

Conservation of energy in the COM frame gives:

$$K^{C\prime} = K^C - Q$$

$$\Rightarrow \frac{1}{2}(2m)v_c'^2 + \frac{1}{2}(2m)v_c'^2 = \frac{3}{8}mv^2 - Q = (\alpha^2 - 1)Q$$

$$\Rightarrow v_c'^2 = (\alpha^2 - 1)\frac{Q}{2m}$$

If the particles scatter with angle Θ in the COM frame, then

$$\vec{v}_{1c}' = v_c'\cos\Theta\,\hat{x} + v_c'\sin\Theta\,\hat{y}$$

$$\vec{v}_{2c}' = -\vec{v}_{1c}' = -v_c'\cos\Theta\,\hat{x} - v_c'\sin\Theta\,\hat{y}$$

Moving back to LAB frame, the velocities of the scattered particles are just the COM velocities $+\vec{V}_{COM}$:

$$\vec{v}_1' = (\frac{1}{4}v + v_c'\cos\Theta)\hat{x} + v_c'\sin\Theta\,\hat{y}$$

$$\vec{v}_2' = (\frac{1}{4}v - v_c'\cos\Theta)\hat{x} - v_c'\sin\Theta\,\hat{y}$$

The angle θ_1 in the LAB frame can be then found as:

$$\tan\theta_1 = \frac{v_{1y}}{v_{1x}} = \frac{v_c'\sin\Theta}{v_c'\cos\Theta + v/4} = \boxed{\frac{\sin\Theta}{\cos\Theta + f(\alpha)}}$$

where

$$f(\alpha) = \frac{v}{4v_c'} = \sqrt{\frac{8\alpha^2 Q}{3m}\frac{2m}{16Q(\alpha^2 - 1)}} = \boxed{\frac{\alpha}{\sqrt{3(\alpha^2 - 1)}}}$$

(c) For $\alpha^2 \gg 1$,

$$f(\alpha) \to \sqrt{\frac{1}{3}} \Rightarrow \tan\theta_1 \to \frac{\sin\Theta}{\cos\Theta + \sqrt{1/3}}$$

For particle 1 to be scattered off at 90°, this requires

$$\tan\theta_1 = \infty \Rightarrow \boxed{\cos\Theta = -\sqrt{\frac{1}{3}}}$$

This is the maximum COM scattering angle for any one of the particles to scatter through 90° in the LAB frame, since it requires $\Theta > 90°$. If the second particle scatters through 90° in the LAB frame:

$$\tan\theta_2 = \frac{\sin\Theta}{\cos\Theta - f(\alpha)} \to \frac{\sin\Theta}{\cos\Theta - \sqrt{1/3}} = \infty \Rightarrow \cos\Theta = \sqrt{\frac{1}{3}}$$

which allows for $\Theta < 90°$ (assuming $\Theta \leq 180°$).

An alternative derivation that doesn't require a solution from part (b) is as follows: Assuming that particle 1 scatters through 90° in the LAB frame, then

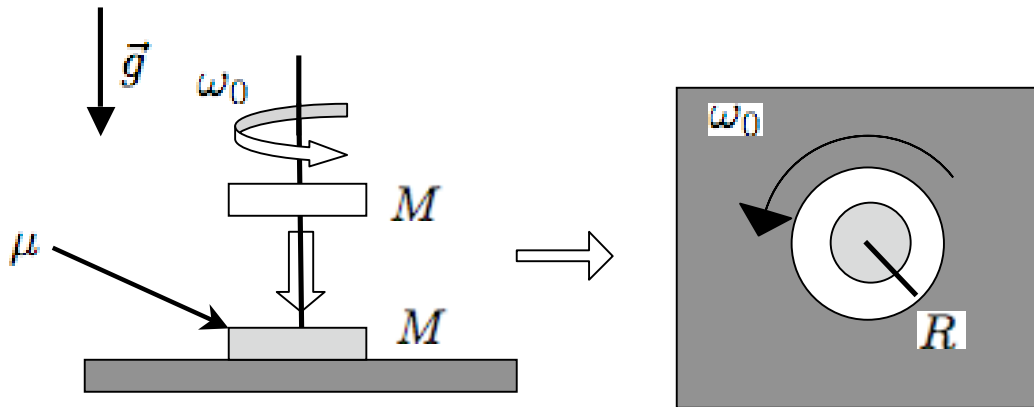$$v_{1x}' = \frac{1}{4}v + v_c'\cos\Theta = 0 \Rightarrow \cos\Theta = -\frac{v}{4v_c'}$$

if $\alpha^2 \gg 1$, then K $\gg$ Q, so in the COM frame:

$$\frac{3}{8}mv^2 = 2mv_c'^2 \Rightarrow \frac{v}{v_c'} = \frac{4}{\sqrt{3}}$$

and hence

$$\cos\Theta = -\sqrt{\frac{1}{3}}$$

Problem 4: Spinning Disk and Hoop [20 pts]

A uniform hoop with mass M, outer radius R and inner radius R/2 is initially
spinning with angular velocity ω_0. The hoop is placed onto a uniform disk, also
of mass M and outer radius R, that is initially at rest on a fixed frictionless table.
The hoop and disk are aligned along their centers of mass. There is a coefficient of
friction μ between the hoop and disk where their surfaces make contact.
Gravitational acceleration is assumed to be constant and points downward.

(a) [10 pts] What is the initial torque (about its center of mass) acting on the disk?
What is the initial torque (about its center of mass) acting on the hoop?

(b) [5 pts] The friction force between the hoop and disk cause them to eventually
spin at the same angular velocity. What is this angular velocity?

(c) [5 pts] How much time does it take for the disk and hoop to reach this common
angular velocity?

SOLUTIONS

(a) The torque on the hoop from the disk arises from friction $\vec{F} = -\mu N \hat{\theta}$, where $N = Mg$ is the normal contact force between the disk and hoop. First consider a small section where the hoop and disk make contact.

$$d\vec{\tau} = \vec{r} \times d\vec{F}$$

$$= r\hat{r} \times -\mu(dm)g\hat{\theta}$$

$$= -r\mu\sigma g(dA)\hat{z}$$

where $dm = \sigma dA = \sigma r dr d\theta$ is a small mass segment on the hoop with uniform areal density σ. The total torque is an integral over the area of the loop:

$$\vec{\tau} = \int d\vec{\tau} = -\int_0^{2\pi} \int_{R/2}^{R} r\mu\sigma g r dr d\theta \hat{z}$$

$$= -\frac{7\pi}{12}\mu\sigma g R^3 \hat{z}$$

but the mass of the hoop is:

$$M = \int_0^{2\pi} \int_{R/2}^{R} \sigma r dr d\theta = \frac{3\pi}{4}\sigma R^2$$

Hence the torque on the hoop as the result of frictional forces from the disk is

$$\boxed{\vec{\tau}_{hoop} = \frac{7}{9}\mu Mg R \hat{z}}$$

Following Newton's third law, the frictional force on the disk from the hoop is equal and opposite to that of the hoop from the disk, and the force acts over the same surface area. Hence, the torques are equal and opposite:

$$\boxed{\vec{\tau}_{disk} = \frac{7}{9}\mu Mg R \hat{z}}$$

(b) There are no external torques on the disk/hoop system, so total angular momentum is conserved, i.e.,

$$I_h \omega_0 = (I_h + I_d)\omega$$

The moment of inertia of the hoop is:

$$I_h = \int_0^{2\pi} \int_{R/2}^R r^2 \sigma r \, dr \, d\theta = \frac{15\pi}{32}\sigma R^4 = \frac{5}{8}MR^2$$

while

$$I_d = \frac{1}{2}MR^2$$

Hence,

$$\omega = \omega_0 \frac{(5/8)MR^2}{(1/2)MR^2 + (5/8)MR^2} = \boxed{\frac{5}{9}\omega_0}$$

(c) The time it takes for the hoop to spin down is given by the torque:

$$\vec{\tau} = \frac{d\vec{L}}{dt} = I_h\dot{\vec{\omega}} \Rightarrow \frac{5}{8}MR^2\dot{\omega} = -\frac{7}{9}\mu M g R$$

$$\Rightarrow \dot{\omega} = -\frac{56}{45}\mu\frac{g}{R}$$

$$\Rightarrow \omega = \omega_0 - \frac{56}{45}\mu\frac{g}{R}t$$

setting ω to the value from part (b) gives:

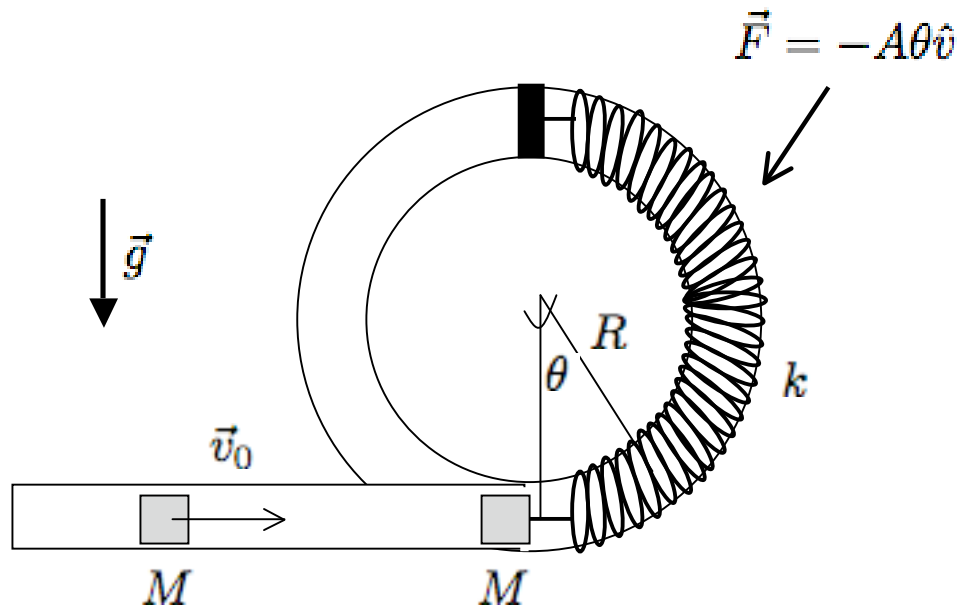$$\boxed{t_{slow} = \frac{5}{14}\frac{R\omega_0}{\mu g}}$$

as a check we can compute the speed up time for the disk:

$$\frac{1}{2}MR^2\dot{\omega} = \frac{7}{9}\mu MgR$$

$$\Rightarrow \omega = \frac{14}{9}\mu\frac{g}{R}t$$

again, substituting the value from part (b) gives:

$$t_{fast} = \frac{5}{14}\frac{R\omega_0}{\mu g}$$

Problem 5: Spring-loaded Tube [20 pts]

$$\vec{F} = -A\theta\hat{v}$$

A block of mass M and velocity v_0 is moving through a fixed tube with frictionless walls. At the bottom of a loop of radius R, the block collides perfectly inelastically (i.e., it sticks) with a plunger of mass M connected to a long fixed spring with spring constant k and equilibrium length πR. As the plunger and block travel around the loop, they are also acted on by a drag force $\vec{F} = -A\theta\hat{v}$; i.e., a force that acts opposite to the direction of motion. Gravitational acceleration acts downward. Assume that the dimensions of the block and plunger are much smaller than R, so that their rotation about their centers of mass are negligible.

(a) [10 pts] What is the incoming block's velocity if the plunger and block come to rest at $\theta = \pi/2$?

(b) [10 pts] Continuing from (a), after the block/plunger pair stop at $\theta = \pi/2$, they swing back around the loop and out into the straight, frictionless linear tube. If there are no adhesive forces that keep the block and plunger bound to each other, at what point does the block move away from the plunger and with what speed?

SOLUTIONS

(a) The initial collision conserves momentum (no external forces acting in short collision period), hence,

$$2Mv_{B+P} = Mv_0 \Rightarrow v_{B+P} = \frac{1}{2}v_0$$

The initial total mechanical energy is then

$$E_{init} = K = \frac{1}{2}(2m)v^2_{B+P} = \frac{1}{4}mv_0^2$$

(Assuming $U = 0$ at the bottom of the loop). At an arbitrary angle θ, we can write down the total mechanical energy as:

$$E(\theta) = K(\theta) + U(\theta) = E_{init} - W(\theta)$$

where U is the potential energy and W the nonconservative work done on the objects. The potential energy arises from gravitational and spring forces:

$$U(\theta) = 2MgR(1 - \cos\theta) + \frac{1}{2}kR^2\theta^2$$

The work is done by the drag force:

$$W = -\int \vec{F}_{nc} \circ d\vec{r} = \int_0^\theta A\theta R d\theta = \frac{1}{2}AR\theta^2$$

At $\theta = \pi/2$, the kinetic energy is:

$$K(\theta) = E_{init} - W(\theta) - U(\theta)$$

if the object is at rest at this point, then

$$E_{init} = W(\theta) + U(\theta)$$

$$\Rightarrow \frac{1}{4}Mv_0^2 = \frac{1}{2}AR\left(\frac{\pi}{2}\right)^2 + 2MgR + \frac{1}{2}kR^2\left(\frac{\pi}{2}\right)^2$$

$$\boxed{\Rightarrow v_0 = \sqrt{\frac{\pi^2}{2}\frac{AR + kR^2}{M} + 8gR}}$$

(b) The block and plunger separate at the bottom of the loop, for at this point the plunger will be pulled back by the spring while the block can move on at the speed

of both objects at that point. At the bottom of the tube, $U = 0$ and the total mechanical energy lost is simply twice the work done by the drag force on the way up:

$$E(0) = E_{init} - 2W\left(\frac{\pi}{2}\right)$$

$$\Rightarrow \frac{1}{2}(2M)v^2 = \frac{1}{4}Mv_0^2 - AR\left(\frac{\pi}{2}\right)^2$$

$$\boxed{\Rightarrow v = \frac{1}{2}\sqrt{v_0^2 - \pi^2\frac{AR}{M}}}$$