8.012 Physics I: Classical Mechanics
Fall 2008

MASSACHUSETTS INSTITUTE OF TECHNOLOGY
Department of Physics

Physics 8.012 Fall 2008

Final Exam

NAME: S O L U T I O N S

Instructions:

1. Do all **SEVEN (7)** problems. You have **2.5 hours**.
2. **Show all work**. Be sure to **CIRCLE YOUR FINAL ANSWER**.
3. Read the questions carefully
4. All work and solutions must be done in the answer booklets provided
5. NO books, notes, calculators or computers are permitted. **A sheet of useful equations is provided on the last page**.

Your Scores

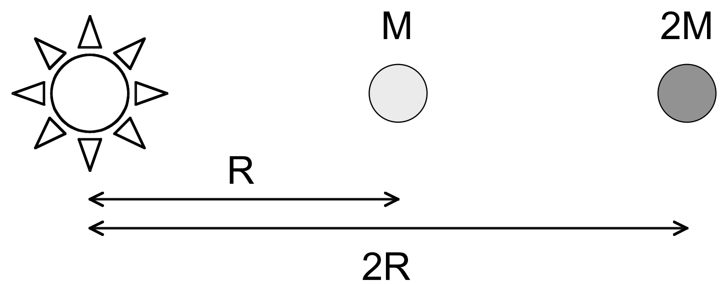
Problem	Maximum	Score	Grader
1	10		
2	15		
3	15		
4	15		
5	15		
6	15		
7	15		
Total	100		

[NO TEST MATERIAL ON THIS PAGE]

Problem 1: Multiple Choice & Short Answer Questions [10 pts]

For each of the following questions enter the correct multiple choice option or write/draw out a short answer in your answer booklet. You do not need to show any work beyond your answer.

(a) [2 pts] Two planets of mass M and 2M are in circular orbits around a star at radii R and 2R, respectively (assume the star's mass is >> M). Which planet has the greater orbital velocity and which planet has the greater orbital angular momentum?



Orbital velocity depends on the strength of the gravitational force, scaling as $v^2 \propto 1/r$; hence, the inner planet moves faster. The angular momentum scales as $mvr \propto r^{1/2}$, so the outer planet (which also has twice the mass) has a greater angular momentum.

(b) [2 pts] What is Chasles' theorem?

(1)	(2)	(3)	(4)
Every force has an equal and opposite pair	Gravitational orbits form ellipses	Motion can be separated in translation of center of mass and rotation about center of mass	Inertial mass equals gravitational mass

(1) is Newton's 3rd law, (2) is Kepler's 1st law, (3) is Chasle's theorem and (4) is the equivalence principle.

(c) [2 pts] A stationary ice skater is spinning about her center of mass (along a principal axis) on a frictionless surface. She pulls in her arms and spins up faster. Which of the following is conserved in this motion (write down all that apply)?

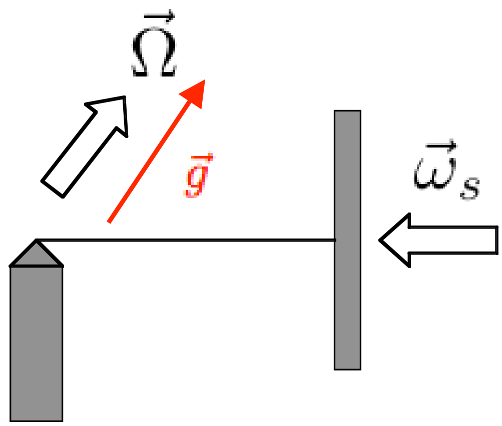
Energy Momentum Angular momentum

Without an external force, the ice skater's momentum doesn't change; similarly, as there are no external torques, angular momentum is conserved. However, rotational energy scales as $L^2/2I$, and the moment of inertia (I) is reduced for the skater as she pulls in her arms, so her total mechanical energy must increase.

(d) [2 pts] What are the dimensions of the gravitational constant G?

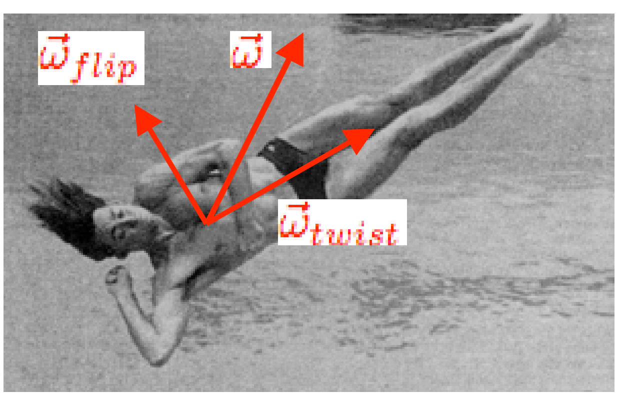
$[M]^{-1}[L]^3[T]^{-2}$

(e) [2 pts] A gyroscope whose spin angular velocity vector points toward the left is observed to precess such that its precession angular velocity vector points at an angle as shown. In which direction does the gravity vector point?



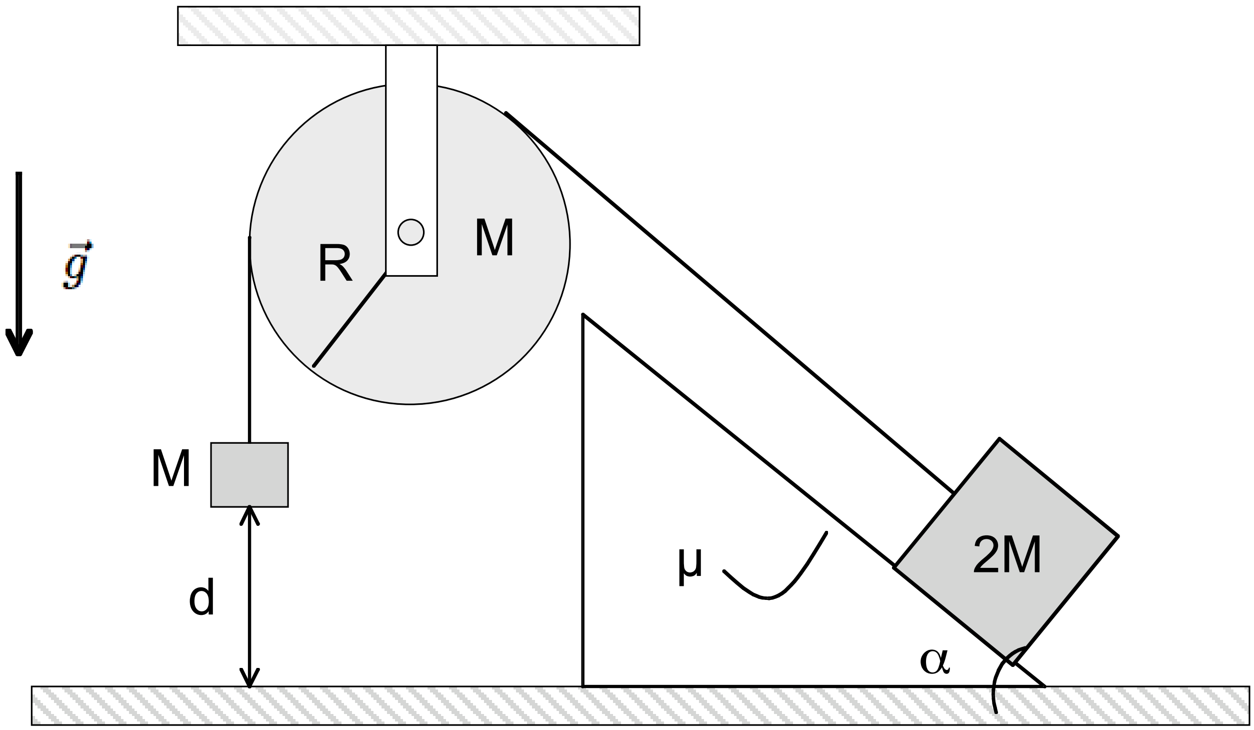
The precession direction points in the opposite direction as the spin vector initially moves toward as the gyroscope falls under gravity. In this case, the gravity vector must therefore be parallel to the precession vector.

(f) [BONUS 2 pts] A diver is the middle of a dive as shown below. Based on clues in the photo, indicate in your answer booklet the direction that his total spin vector points, and determine whether the diver is doing a front flip or a back flip.



One clue is the hair, which lies in a plane perpendicular to the spin vector (twirl a handful of string to convince yourself of this). The placement of the arms breaks the degeneracy, indicating an applied torque that causes a twist rotation whose direction points along the feet. So the total spin vector points in the direction shown, and the flip component indicates a front flip (head over feet).

Problem 2: Atwood Machine [15 pts]

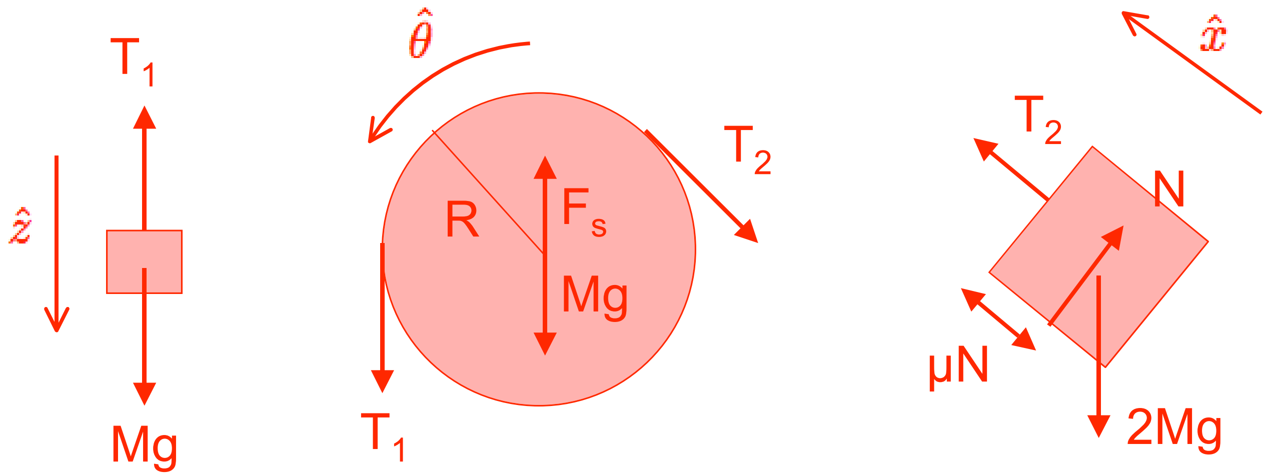


An Atwood machine consists of a fixed pulley wheel of radius R and uniform mass M (a disk), around which an effectively massless string passes connecting two blocks of mass M and 2M. The lighter block is initially positioned a distance d above the ground. The heavier block sits on an inclined plane with opening angle α. There is a coefficient of friction μ between the surfaces of this block and the inclined plane. Constant gravitational force acts downwards, and assume that the string never slips.

(a) [5 pts] Determine two conditions on the angle α which allow the lighter block to move up or move down.

(b) [10 pts] Assuming that the lighter block moves down, determine its acceleration.

SOLUTION TO PROBLEM 2



(a) The above diagram shows the appropriate forces on the two blocks and the pulley wheel. The two conditions for the blocks arise from whether the leftmost block moves up or down, which changes the direction of the friction force acting on the rightmost block. Consider first the leftmost block moving down; in that case $Mg > T_1$ and $T_1 > T_2$ (so the wheel can spin), and T_2 must be greater than both the friction force ($\mu N = \mu 2Mg\cos\alpha$) and the component of gravitational force parallel to the inclined plane surface ($2Mg\sin\alpha$) on the rightmost block:

$$Mg > T_1 > T_2 > 2Mg\sin\alpha + \mu 2Mg\cos\alpha$$

$$\Rightarrow \boxed{\sin\alpha + \mu\cos\alpha < \frac{1}{2}}$$

If the leftmost block moves up, then $T_1 > Mg$, $T_2 > T_1$ and the gravitational force on the rightmost block must overcome both tension and friction:

$$2Mg\sin\alpha > T_2 + \mu 2Mg\cos\alpha$$

$$\Rightarrow Mg < T_1 < T_2 < 2Mg\sin\alpha - \mu 2Mg\cos\alpha$$

$$\Rightarrow \boxed{\sin\alpha - \mu\cos\alpha > \frac{1}{2}}$$

(b) Choosing our coordinate systems for each component as shown above so that all objects move in a positive direction, we can write down the following equations of motion:

leftmost block: $M\ddot{z} = Mg - T_1$

pulley wheel: $I\ddot{\theta} = \frac{1}{2}MR^2\ddot{\theta} = RT_1 - RT_2$

rightmost block: $2M\ddot{z} = T_2 - 2Mg\sin\alpha - \mu 2Mg\cos\alpha$

The constraint equation tying all of these objects together (connected by an massless and hence inextensible string) is:

$\ddot{z} = R\ddot{\theta} = \ddot{x}$

Using this and the first two equations of motion we can relate to two tension forces:

$$T_1 = \frac{2}{3}T_2 + \frac{1}{3}Mg$$

and using the first and third equations of motion we can solve for the individual tensions:

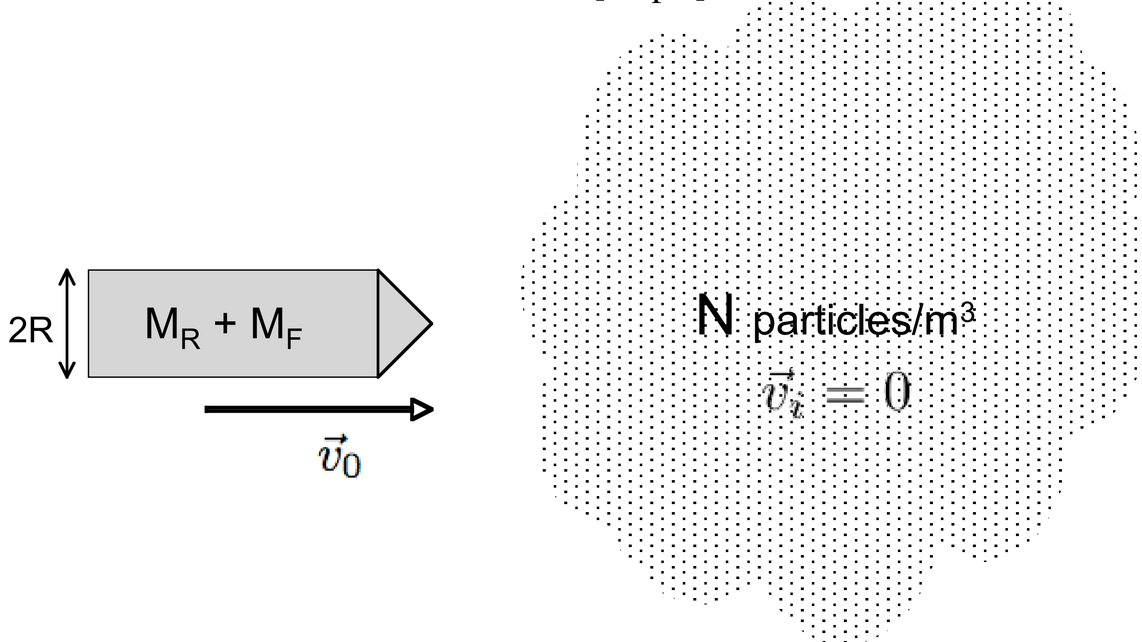
$$T_2 = Mg(\frac{4}{7} + \frac{6}{7}\sin\alpha + \frac{6}{7}\mu\cos\alpha)$$

$$T_1 = Mg(\frac{5}{7} + \frac{4}{7}\sin\alpha + \frac{4}{7}\mu\cos\alpha)$$

$$\Rightarrow \ddot{z} = g - \frac{T_1}{M} = \boxed{g(\frac{2}{7} - \frac{4}{7}\sin\alpha - \frac{4}{7}\mu\cos\alpha)}$$

Note that the first conditions from part (a) is necessary for acceleration to be positive.

Problem 3: Rocket in an Interstellar Cloud [15 pts]



A cylindrical rocket of diameter 2R, mass M_R and containing fuel of mass M_F is coasting through empty space at velocity v_0. At some point the rocket enters a uniform cloud of interstellar particles with number density N (e.g., particles/m^3), with each particle having mass m ($\ll M_R$) and initially at rest. To compensate for the dissipative force of the particles colliding with the rocket, the rocket engines emit fuel at a rate $dm/dt = \gamma$ at a constant velocity u with respect to the rocket. Ignore gravitational effects between the rocket and cloud particles.

(a) [5 pts] Assuming that the dissipative force from the cloud particles takes the form $F = -Av^2$, where A is a constant, derive the equation of motion of the rocket (F = ma) through the cloud as it is firing its engines.

(b) [5 pts] What must the rocket's thrust be to maintain a constant velocity v_0?

(c) [5 pts] If the rocket suddenly runs out of fuel, what is its velocity as a function of time after this point?

(d) [BONUS 5 pts] Assuming that each cloud particle bounces off the rocket elastically, and collisions happen very frequently (i.e., collisions are continuous), prove that the dissipative force is proportional to v^2, and determine the constant A. Assume that the front nose-cone of the rocket has an opening angle of 90°.

SOLUTION TO PROBLEM 3:

(a) Consider some time t when a parcel of mass dm is ejected from the rocket at velocity u. Newton's second law can be written as (assuming the rightward direction to be positive):

$$\frac{dp}{dt} = F \Rightarrow p(t+dt) - p(t) = F\,dt = -Av^2 dt$$

The momentum of the rocket + fuel system before and after ejection of fuel can be written as:

$$p(t) = (M + dm)v$$
$$p(t+dt) = M(v+dv) + dm(v+dv-u)$$

Keeping only first order terms we derive

$$M\,dv - u\,dm = -Av^2 dt$$
$$\Rightarrow \boxed{M\frac{dv}{dt} = \gamma u - Av^2}$$

or

$$\boxed{(M_R + M_f - \gamma t)\frac{dv}{dt} = \gamma u - Av^2}$$

where γ has been substituted in place of dm/dt.

(b) To maintain constant speed, dv/dt = 0 hence the thrust is

$$\boxed{\gamma u = Av_0^2}$$

(c) When the rocket runs out of fuel, it has a mass M_R and there is no thrust term, hence the equation in part (a) becomes:

$$M_R\frac{dv}{dt} = -Av^2$$

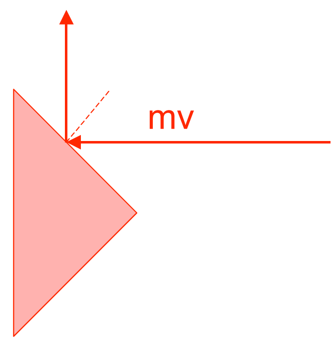
This equation is separable and can be directly integrated:

$$\int_{v_0}^{v(t)} \frac{dv}{v^2} = -\int_0^t \frac{A}{M_R} dt$$

$$\Rightarrow \boxed{\frac{1}{v(t)} - \frac{1}{v_0} = \frac{A}{M_R} t}$$

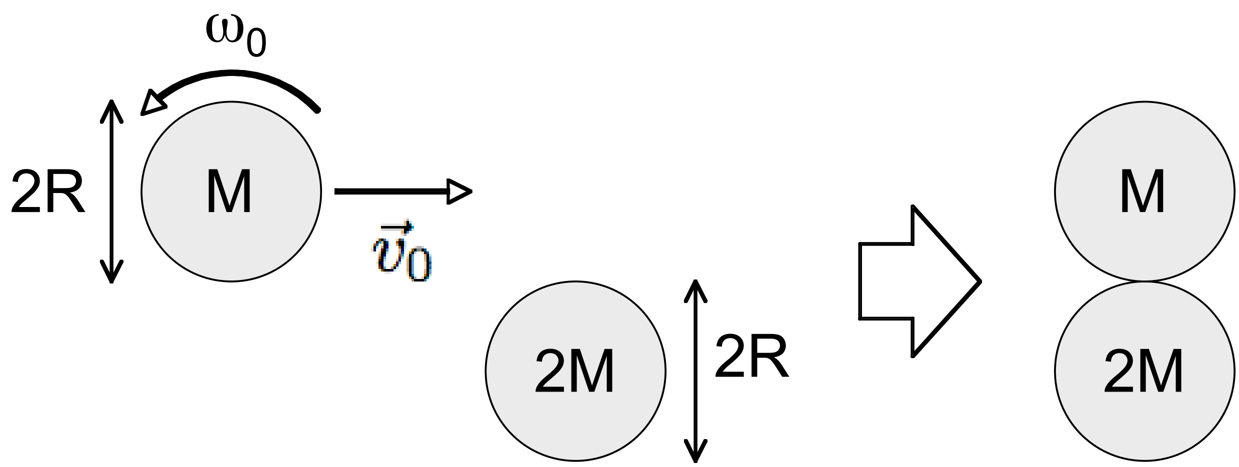
$$\Rightarrow \boxed{v(t) = \frac{v_0}{1 + \frac{Av_0}{M_R} t}}$$

(d) As illustrated in the figure to the right, each particle that collides with the rocket is deflected through 90° (due to geometry), which means that each particle imparts an impulse on the rocket of Δp = mv in the horizontal direction opposite of motion (it also imparts an impulse of mv in the vertical direction, but that is balanced by particles striking the other side of the nosecone). The number of particles that strikes the rocket per unit time is simple the volume swept through by the rocket per unit time, AΔx/Δt = $\pi R^2 v$. The total momentum transfer onto the rocket is:



$$\frac{dp}{dt} = -N(mv)\pi R^2 v = \boxed{-Nm\pi R^2 v^2}$$

Problem 4: Sticky Disks [15 pts]



A uniform disk of mass M and diameter 2R moves toward another uniform disk of mass 2M and diameter 2R on the surface of a frictionless table. The first disk has an initial velocity v_0 and spin rate ω_0 as indicated, while the second disk is initially stationary. When the first disk contacts the second (a "glancing" collision), they instantly stick to each other and move as a single object.

(a) [5 pts] What are the velocity and spin angular velocity of the combined disks after the collision? Indicate both magnitudes and directions.

(b) [5 pts] For what value of ω_0 would the combined disks not rotate?

(c) [5 pts] How much total mechanical energy is lost in this collision assuming that the combined disk system is not rotating?

SOLUTION TO PROBLEM 4:

(a) There are no external forces on this system, so the initial and final momenta are the same and equal to Mv_0 toward the right. The total mass of the combined disk system is 3M, so the final velocity is

$$\boxed{\vec{v} = \frac{1}{3}\vec{v}_0}$$

The total angular momentum of the system is also conserved, but in this case we must be more careful, as the question asks for the final spin angular velocity, so it is important to calculate the initial and final angular momenta about the center of mass of the combined disk system. This position lies at a distance:

$$R_{COM} = \frac{RM - R(2M)}{3M} = -\frac{1}{3}R$$

below the contact point of the two disks. About this point, the initial angular momentum of the system comes from both rotation of the moving disk and its center of mass translation with respect to the center of mass of the combined disk system:

$$\vec{L}_{tot} = \frac{1}{2}MR^2\omega_0\hat{\odot} - \frac{4}{3}RMv_0\hat{\odot} = I_f\vec{\omega}_f$$

The final moment of inertia of the two disks can be found by first adding the center of mass moments of inertia of the two disks separately and thn applying the parallel axis theorem (moving to the center of mass of the combined disk system):

$$I_f = \frac{1}{2}MR^2 + M(\frac{4}{3}R)^2 + \frac{1}{2}(2M)R^2 + 2M(\frac{2}{3}R)^2 = \frac{25}{6}MR^2$$

Hence the final spin angular velocity is

$$\boxed{\vec{\omega}_f = (\frac{3}{25}\omega_0 - \frac{8}{25}\frac{v_0}{R})\hat{\odot}}$$

(b) For the final spin angular velocity to be zero, the two terms above cancel, so

$$\boxed{\omega_0 = \frac{8}{3}\frac{v_0}{R}}$$

(c) To keep the math clean, it is assumed that the final system is not spinning, so that ω_0 and v_0 are related as in part (b). The initial energy is therefore:

$$E_i = \frac{1}{2}Mv_0^2 + \frac{1}{2}I\omega_0^2 = \frac{1}{2}Mv_0^2 + \frac{1}{2}(\frac{1}{2}MR^2)(\frac{8}{3}\frac{v_0}{R})^2 = \frac{41}{18}Mv_0^2$$

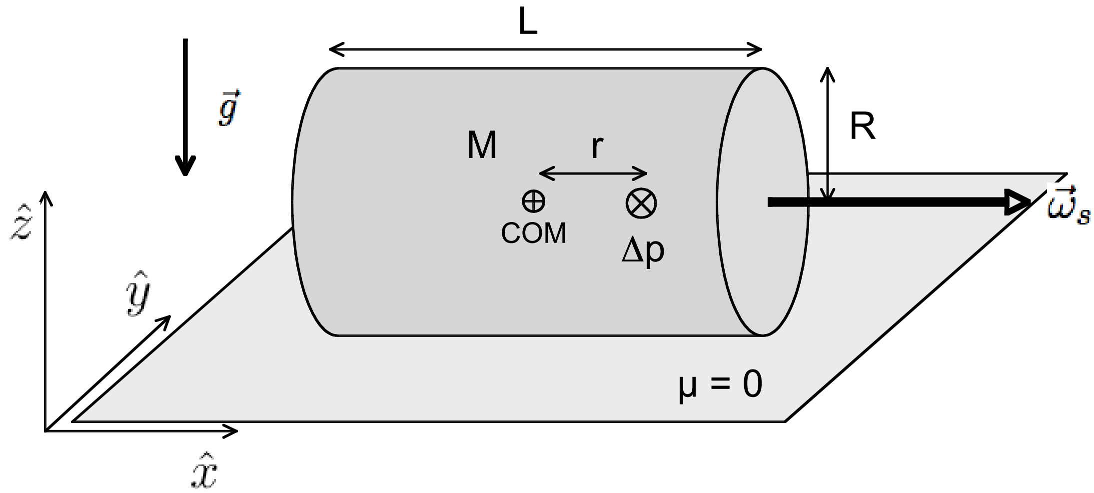
The final energy is just the translation energy:

$$E_f = \frac{1}{2}(3M)(\frac{v_0}{3})^2 = \frac{1}{6}Mv_0^2$$

so the total energy loss is:

$$E_f - E_i = (\frac{1}{6} - \frac{41}{18})Mv_0^2 = \boxed{-\frac{19}{9}Mv_0^2}$$

Problem 5: Cylindrical Top [15 pts]



A cylinder of mass M, length L and radius R is spinning about its long axis with angular velocity $\vec{\omega} = \omega_s \hat{x}$ on a frictionless horizontal surface. The cylinder is given a sharp, horizontal strike with impulse $\Delta p\,\hat{y}$ at a distance r from its center of mass (COM). Assume that constant gravitational acceleration acts downward. NOTE: you do not need to use Euler's equations to solve this problem.

(a) [5 pts] What is the translational velocity of the cylinder after the impulse (magnitude and direction)?

(b) [5 pts] The strike imparts an angular momentum impulse to the cylinder which causes it to lift up at one end. At what angle α will the cylinder be tilted after the impulse and which end of the cylinder lifts up? Assume that the angular momentum impulse is much smaller than the spin angular momentum.

(c) [5 pts] After the cylinder tilts up, it effectively becomes a top. Determine its precessional rate and the direction of precession. Assume that nutational motion is negligible (i.e., α remains effectively constant) and that R << L (i.e., that the cylinder can be approximated as a thin rod for this part).

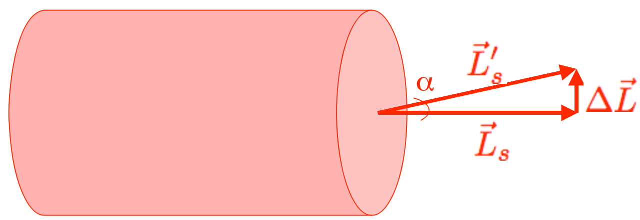
(d) [5 pts BONUS] For a strong enough impulse, the cylinder will tilt high enough to precess in the opposite direction. What is the minimum tilt angle for this to happen and what is the minimum impulse required? (Note that you cannot assume R << L here. This problem is similar to the "tipping battery" trick pointed out by one of the 8.012 students.)

SOLUTION TO PROBLEM 5:

(a) The impulse provides the only external force to the system, so the total momentum of the cylinder is simply the impulse, Δp in the y-direction. Hence, the translational velocity of the cylinder is:

$$\boxed{\vec{v} = \frac{\Delta p}{M}\hat{y}}$$

(b) The angular impulse imparted on the cylinder about its center of mass is

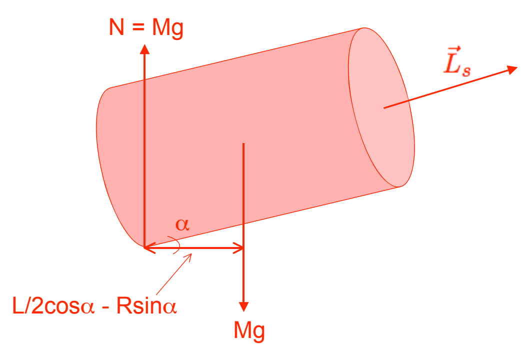


$$\Delta\vec{L} = r\hat{x}\times\Delta p\hat{y} = r\Delta p\hat{z}$$

This impulse adds vectorally to the spin angular momentum of the cylinder, as shown in the figure to the right. If we assume that $L_S >> \Delta L$ (our standard gyroscopic approximation), then the magnitude of the total angular momentum is still equal to $I\omega_S$ but points in a direction offset by an angle α upwards. Since this is still the cylinder spinning about its axis, it must be that the right side of the cylinder tips up by an angle α:

$$\tan\alpha \approx \alpha = \frac{\Delta L}{L_s} = \boxed{\frac{r\Delta p}{I\omega_s} = \frac{2r\Delta p}{MR^2\omega_s}}$$

(c) With the cylinder tipped up, gravity and the normal contact force on the ground now exert a net torque on the cylinder, which will cause the spin angular momentum vector to precess with rate Ω. Let's measure angular momenta and torque about the center of mass of the cylinder (alternately we could have measured about the pivot point, but since N=Mg, the result is the same). The horizontal lever arm between the pivot point and the center of mass is (L/2)cosα - Rsinα, but assuming L >> R we can drop the second term. The torque is then:



$$\vec{\tau} = (-\frac{L}{2}\cos\alpha\hat{x}) \times (Mg\hat{z}) = \frac{L}{2}Mg\cos\alpha\hat{y} = \vec{\Omega}\times\vec{L}_s$$

using here our expression for the time derivative of a vector (L_S) in a rotating reference frame. The precession vector must point along the z-direction, a fact we can ascertain by considering that gravity would initially pull the spin angular momentum vector down, so to conserve total angular momentum the precession angular momentum vector must point upward. The precession vector rotates only the radial component of the spin angular momentum vector, hence:

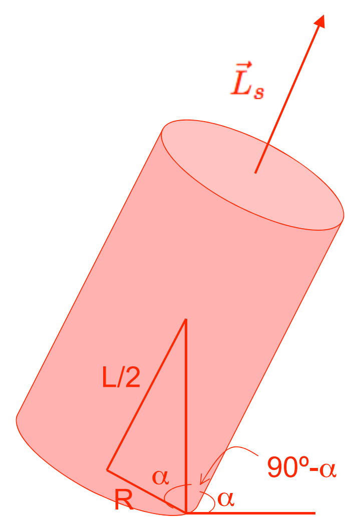
$$\vec{\Omega}\times\vec{L}_s = \Omega L_S \cos\alpha\hat{y}$$

Solving for the precession rate:

$$\vec{\Omega} = \boxed{\frac{MLg}{2I\omega_s}\hat{z} = \frac{Lg}{R^2\omega_s}\hat{z}}$$

note that the I here is the moment inertia about the spinning axis, not about the precession axis.

(d) To precess in the other direction, the center of mass must be inside the pivot point of the disk, which happens at a critical angle (see right):



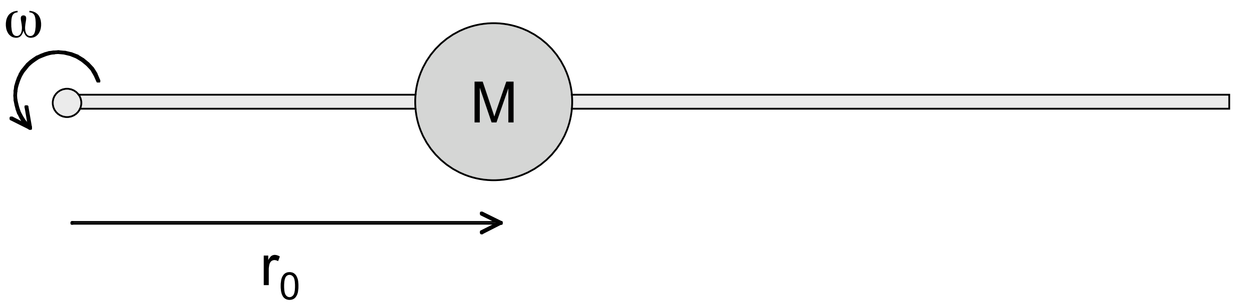
$$\tan\alpha > \frac{L}{2R}$$

from the solution to (a) this places a constraint on the impulse required:

$$\boxed{\Delta p > \frac{MRL\omega_s}{4r}}$$

Note that this is approximate, as we no longer satisfy the gyroscopic approximation that $L_S >> \Delta L$ (indeed, they are of the same order of magnitude in this case).

Problem 6: Bead on a Spinning Rod [15 pts]



A bead of mass M is placed on a frictionless, rigid rod that is spun about at one end at a rate ω. The bead is initially held at a distance r_0 from the end of the wire. For the questions below, treat the bead as a point mass. Ignore gravitational forces.

(a) [5 pts] What force is necessary to hold the bead in place at r_0? Indicate both magnitude and direction.

(b) [5 pts] After the bead is released, what is its position in the inertial frame (in polar coordinates) as a function of time?

(c) [5 pts] Now calculate the fictitious forces on the bead in a reference frame that is rotating with the wire. What real force must the rod exert on the bead in both the rotating and inertial frames?

SOLUTION TO PROBLEM 6:

(a) The force applied to hold the bead in place is simply the centripetal force:

$$\vec{F} = -\frac{mv^2}{r} = \boxed{-mr_0\omega^2\hat{r}}$$

(b) In the inertial frame, we can write down the equations of motion in polar coordinates assuming that there is no radial force acting (no friction or constraint force):

$$m(\ddot{r} - r\dot{\theta}^2) = 0$$

$$m(r\ddot{\theta} + 2\dot{r}\dot{\theta}) = N$$

Here, N is some normal force acting on the bead in the angular direction (important for part c). The angular position is straightforward, since the rod is rotating at constant angular rate, hence,

$$\boxed{\theta(t) = \theta_0 + \omega t}$$

For the radial position, we must solve the radial equation of motion:

$$\ddot{r} = \omega^2 r$$

This can be done by trial, or you can simply remember that the general solution for this equation is:

$$r(t) = Ae^{\omega t} + Be^{-\omega t}$$

Using the initial conditions $r(0) = r_0$ and $dr/dt(0) = 0$, we find:

$$A + B = r_0$$

$$\omega(A - B) = 0 \Rightarrow A = B$$

So

$$\boxed{r(t) = \frac{r_0}{2}(e^{\omega t} + e^{-\omega t})}$$

(c) In the rotating reference frame, there is only radial motion, and there are two fictitous forces acting: centrifugal and coriolis forces:

$$\vec{F}_{cf} = -m(\omega\hat{z} \times \omega\hat{z} \times r\hat{r}) = m\omega^2 r\hat{r}$$

$$\vec{F}_{cor} = -m(\omega\hat{z} \times \dot{r}\hat{r}) = -m\omega\dot{r}\hat{\theta}$$

The angular equation of motion in the rotating frame is:

$$m(r\ddot{\theta} + 2\dot{r}\dot{\theta}) = N - m\omega\dot{r} = 0$$

where N is again the angular normal force acting on the bead, and the net angular acceleration component is 0 since θ is constant in the rotating frame. Hence

$$\vec{N} = m\omega\dot{r}\hat{\theta} = \boxed{\frac{m\omega^2 r_0}{2}(e^{\omega t} - e^{-\omega t})\hat{\theta}}$$

Problem 7: Central Potential [15 pts]

A particle of mass m moves within a region under the influence of a force of the form

$$\vec{F}(r) = -Ar^3\hat{r}$$

The particle is initially at a distance r_0 from the origin of the force, and initially moves with velocity v_0 in a tangential direction.

(a) [5 pts] Derive and sketch the effective potential of this system as a function of radius from the origin. Indicate all important inflection points. Can the particle pass through the origin of this reference frame?

(b) [5 pts] Find the velocity v_0 required for the particle to move in a purely circular orbit at a radius r_0 with this force law.

(c) [5 pts] Compute the frequency of small oscillations about this equilibrium radius. How does the period of these oscillations compare to the orbital period?

SOLUTION TO PROBLEM 7:

(a) The effective potential arises from the radial equation of motion assuming that the total angular momentum is a constant:

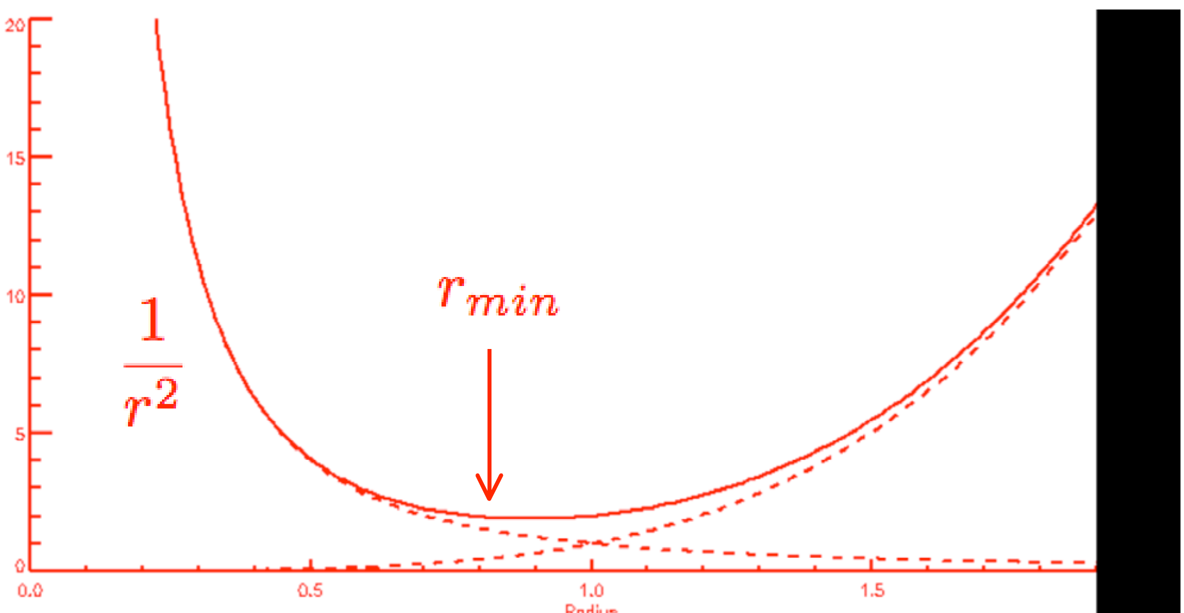
$$m\ddot{r} = F(r) + mr\dot{\theta}^2 = -Ar^3 + \frac{l^2}{mr^3}$$

where the angular momentum is defined as

$$l = mvr = mv_0 r_0$$

The potential arising from this total force law is

$$U_{eff} = -\int F_{tot}\,dr = \boxed{\frac{A}{4}r^4 + \frac{l^2}{2mr^2} + \text{constant}}$$



The figure above provides a rough sketch of this function (with A = 4 and $l^2 = 2m$). There is one minimum inflection (equilibrium) point where the net force vanishes:

$$\frac{l^2}{mr_{min}^3} = Ar_{min}^3$$

$$\Rightarrow \boxed{r_{min} = \left(\frac{l^2}{mA}\right)^{1/6} = \left(\frac{mv_0^2 r_0^2}{A}\right)^{1/6}}$$

This is a stable equilbrium point. The potential tends to infinity as r → 0, so it is not possible to pass through the origin with this potential.

(b) For a purely circular orbit, the object must reside at its minimum radius in the potential, hence:

$$r_0 = \left(\frac{mv_0^2 r_0^2}{A}\right)^{1/6}$$

$$\Rightarrow \boxed{v_0 = \sqrt{\frac{Ar_0^4}{m}}}$$

(c) The frequency of small oscillations in any potential can be derived from the second derivative of the potential about its (stable) equilibrium point:

$$\omega^2 = \frac{1}{m}\frac{d^2U_{eff}}{dr^2}|_{r_{min}} = -\frac{1}{m}\frac{dF}{dr}|_{r_{min}}$$

$$= 3\frac{Ar_{min}^2}{m} + \frac{3l^2}{m^2 r_{min}^4}$$

$$= 6\left(\frac{r_0 v_0 A}{m}\right)^{2/3}$$

Substituting in our expression for v_0 from part (b) reduces this to:

$$\boxed{\omega^2 = 6\frac{Ar_0^2}{m}}$$

The period of this oscillation is:

$$P_{osc} = \frac{2\pi}{\omega} = 2\pi\sqrt{\frac{m}{6Ar_0^2}}$$

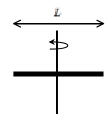
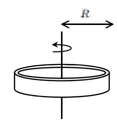
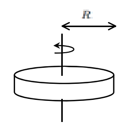
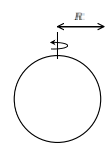
The period of rotational motion is:

$$P_{rot} = \frac{2\pi r_0}{v_0} = 2\pi\sqrt{\frac{m}{Ar_0^2}}$$

Hence, the ratio of these periods is $\sqrt{6}$.

USEFUL EQUATIONS

Velocity in polar coordinates	$\dot{\vec{r}} = \dot{r}\hat{r} + r\dot{\theta}\hat{\theta}$
Acceleration in polar coordinates	$\ddot{\vec{r}} = (\ddot{r} - r\dot{\theta}^2)\hat{r} + (r\ddot{\theta} + 2\dot{r}\dot{\theta})\hat{\theta}$
Center of mass (COM) of a rigid body	$\vec{R} = \frac{\sum m_i \vec{r}_i}{\sum m_i} = \frac{1}{M}\int \rho \vec{r} dV$
Volume element in cylindrical coordinates	$dV = r dr d\theta dz$
Kinetic energy	$K = \frac{1}{2}M(\vec{v}\cdot\vec{v}) + \frac{1}{2}\vec{\omega}\cdot\mathbf{I}\cdot\vec{\omega}$
Work	$W = \Delta K = \int \vec{F}\cdot d\vec{r} = \int \vec{\tau}\cdot d\vec{\theta}$
Potential Energy (for conservative forces)	$U = -\int \vec{F}_c \cdot d\vec{r}$ where $\vec{\nabla}\times\vec{F}_c = 0$
Angular momentum	$\vec{L} = \vec{r}\times\vec{p} = \mathbf{I}\cdot\vec{\omega}$
Torque	$\vec{\tau} = \vec{r}\times\vec{F} = \frac{d\vec{L}}{dt}$ Fixed axis rotation: $\tau_z = I_{zz}\dot{\omega}$

COM Moment of inertia for a uniform bar	 $I_{zz} = \frac{1}{12}ML^2$
COM Moment of inertia for a uniform hoop	 $I_{zz} = MR^2$
COM Moment of inertia for a uniform disk	 $I_{zz} = \frac{1}{2}MR^2$
COM Moment of inertia for a uniform sphere	 $I_{zz} = \frac{2}{5}MR^2$
Scalar parallel axis theorem	$I = I_{COM} + MR^2$
Moments of inertia tensor (permute x→y→z)	$I_{xx} = \sum_i m_i(y_i^2 + z_i^2) = \int dV \rho(y^2 + z^2)$
Euler's Equations (permute 1→2→3)	$\tau_1 = I_1\dot{\omega}_1 + (I_3 - I_2)\omega_2\omega_3$
Time derivative between inertial and rotating frames	$\left(\frac{d\vec{B}}{dt}\right)_{inertial} = \left(\frac{d\vec{B}}{dt}\right)_{rotating} + \vec{\Omega} \times \vec{B}$
Fictitious force in an accelerating frame	$\vec{F}_f = -m\vec{A}$
Fictitious force in a rotating frame (Ω constant)	$\vec{F}_f = -2m\vec{\Omega} \times \vec{v}_{rot} - m\vec{\Omega} \times (\vec{\Omega} \times \vec{r})$
Taylor Expansion of f(x)	$f(x) = f(a) + \frac{1}{1!}\frac{df}{dx}\|_a(x-a) + \frac{1}{2!}\frac{d^2f}{dx^2}\|_a(x-a)^2$