8.012 Physics I: Classical Mechanics
Fall 2008

Physics 8.012   Fall 2008

Exam 2

SOLUTIONS

NAME: _____

MIT ID number: _____

Instructions:

1. Do all **FIVE (5)** problems. You have **90 minutes**.
2. **SHOW ALL WORK**. Be sure to **circle your final answer**.
3. Read the questions carefully.
4. All work must be done in this booklet in workspace provided. Extra blank pages are provided at the end if needed.
5. NO books, notes, calculators or computers are permitted. **A sheet of useful equations is provided on the last page**.
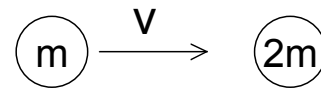
Your Scores

Problem	Maximum	Score	Grader
1	10		
2	20		
3	20		
4	25		
5	25		
Total	100		

Problem 1: Quick Multiple Choice Questions [10 pts]

For each of the following questions circle the correct answer. You do not need to show any work.

(a) Which of the following is the result of an elastic collision between the two isolated objects at right?

$$\text{(m)} \xrightarrow{\ \ V\ \ } \text{(2m)}$$

$\xleftarrow{\ \ V\ \ }$ (m) (2m)	(m) (2m) $\xrightarrow{\ \ v/2\ \ }$
$\xleftarrow{\ \ v/3\ \ }$ (m) (2m) $\xrightarrow{\ \ 2v/3\ \ }$	$\xleftarrow{\ \ V\ \ }$ (m) (2m) $\xrightarrow{\ \ V\ \ }$

This can be verified by examining conservation of momentum and energy (since it is an elastic collision).

(b) Which experiences the **largest** impulse?

| Fully elastic collision of 10 kg ball dropped from 10 m	Fully inelastic collision of 20 kg ball dropped from 10 m	Fully elastic collision of 20 kg ball dropped from 5m	These all have the same impulse

The first two choices experience the same impulse, which equals (1 or 2) $m\sqrt{gh}$.

The 3$^{\text{rd}}$ option experiences the largest impulse by a factor of $\sqrt{2}$

(c) A cart traveling at speed v on a frictionless track starts to leak sand. What is the cart's speed at a later time?

Greater than v	Less than v	Equal to v

The sand experiences no force from the cart as it leaves, so it applies no force on the cart and cannot change the cart's momentum.

(d) Consider the composite disk at right, with a main section of mass density ρ and radius d, and a subsection of mass density 2ρ and radius d/2. If O marks the origin of our coordinate system, at the center of the composite disk, where is the disk's center of mass located?
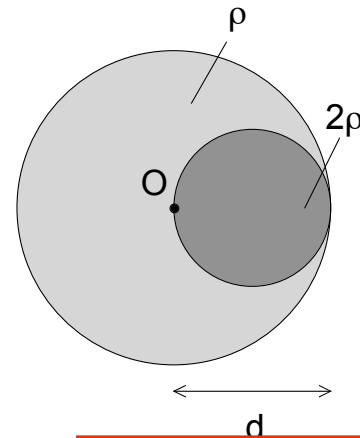
| at O	d/2 to right of O	d/6 to right of O	d/10 to right of O

This can be determined using the center of mass formula, adding together the centers of mass of two solid disks with density ρ (superposition).

(e) A block connected to a fixed wall by a spring oscillates back and forth on a frictionless surface. When it reaches its **maximum extent**, a piece of clay is dropped onto it and instantaneously sticks. Which of the following is conserved in this collision?
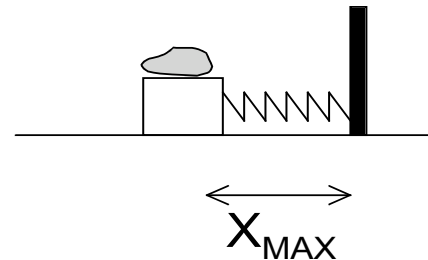
Horizontal momentum	Total mechanical energy	Total mechanical energy and horizontal momentum	None of these

When the clay sticks, in comes to rest so its mechanical energy is not conserved. However, the collision does not affect its horizonal momentum, which is 0, nor that of the block.

Problem 2: Collision [20 pts]
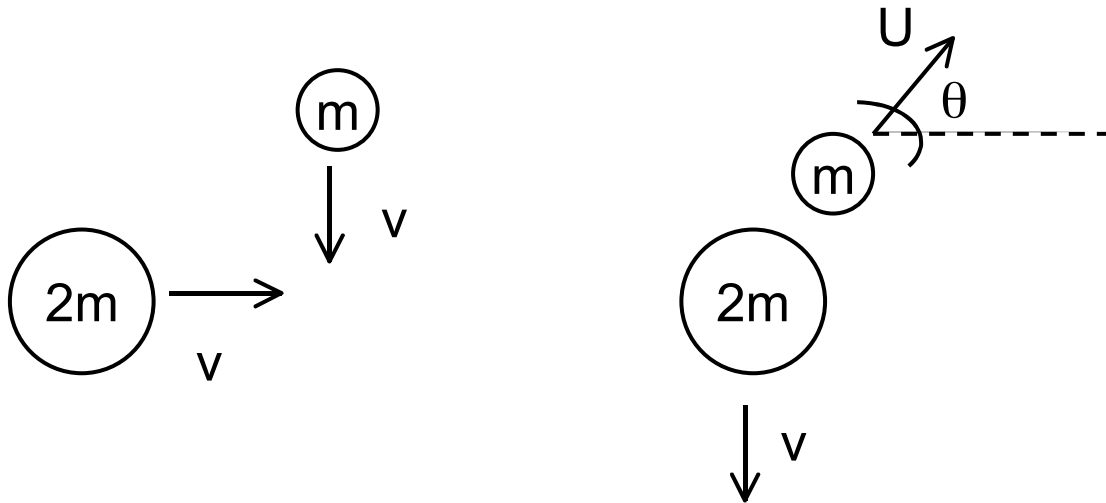
Two balls, one of mass m and one of mass 2m, approach from orthogonal directions with identical speeds v and collide. After the collision, the more massive ball moves with the same speed v but downward (orthogonal to its original direction) and the less massive ball moves with speed U at an angle θ with respect to horizontal. Assume that no external forces act during the collision.

(a) [10 pts] Calculate the final speed U of the less massive ball and the angle θ.

(b) [10 pts] Determine how much kinetic energy is lost or gained by the two balls during the collision. Is this collision elastic, inelastic or superelastic?

[SOLUTION]

(a) Momentum in both horizontal (x) and vertical (y) directions must be conserved, hence:

$$\hat{x} : 2mv = mU\cos\theta$$

$$\hat{y} : mv = 2mV - mU\sin\theta$$

so:

$$U\cos\theta = 2v$$

$$U\sin\theta = v$$

so
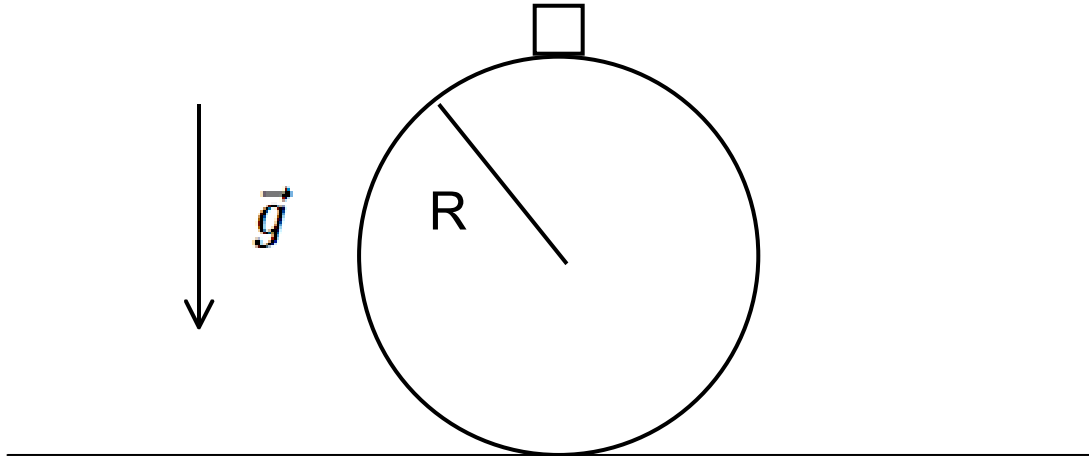
$$\boxed{\begin{aligned}\tan\theta &= \frac{1}{2} \\ U &= \frac{v}{\sin\theta} = \sqrt{5}v\end{aligned}}$$

(b) The change in kinetic energy is just the change in the kinetic energy of the small mass (since the big mass has the same KE before and after the collision):

$$\Delta K = \frac{1}{2}mU^2 - \frac{1}{2}mv^2 = \frac{1}{2}m(5v^2 - v^2) = \boxed{2mv^2}$$

the collision is superelastic.

Problem 3: Cube on a Sphere [20 pts]

A small block starts from rest and slides down from the top of a fixed sphere of radius R, where R >> size of the block. The surface of the sphere is frictionless and constant gravitational acceleration g acts downward.

(a) [10 pts] Determine the speed of the block as a function of angle from the top while it remains in contact with the sphere.

(b) [10 pts] At what angle does the block lose contact with the sphere?

[SOLUTION]

(a) Only conservative forces are present, so total mechanical energy is conserved. Hence the KE of the block comes simply from extracting gravitational potential energy:

$$\frac{1}{2}mv^2 = mg\Delta h = mg(2R - R(1 + \cos\theta)) = mgR(1 - \cos\theta)$$
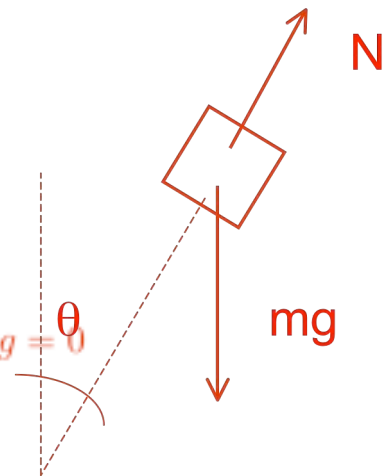
$$\Rightarrow \boxed{v = \sqrt{2gR(1 - \cos\theta)}}$$

(b) The block loses contact when the normal force acting on it disappears. Using polar coordinates and the force diagram at right, we can write down the radial equation of motion:
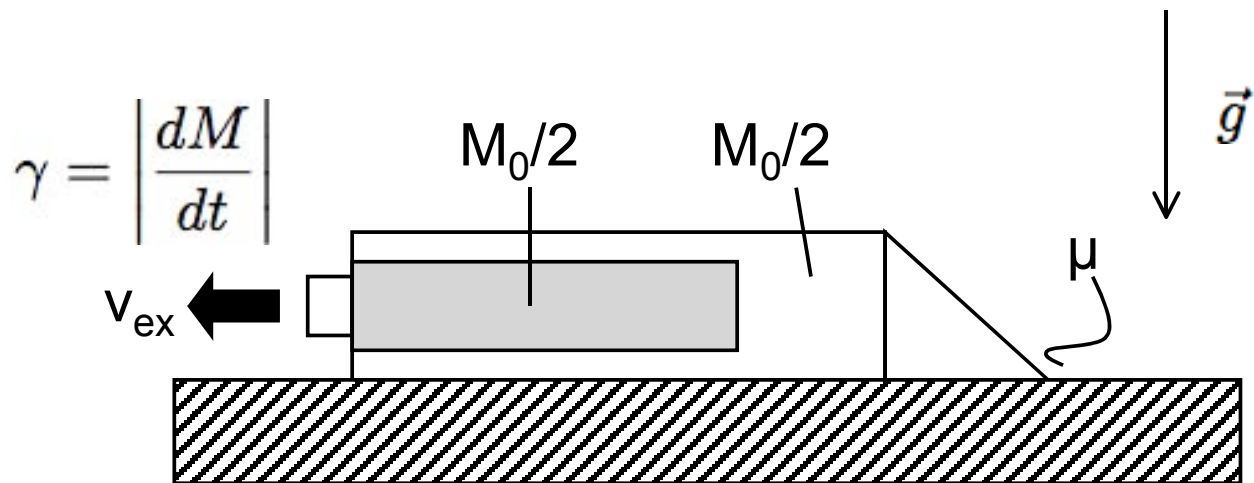
$$m(\ddot{r} - r\dot{\theta}^2) = -\frac{mv^2}{r} = N - mg\cos\theta$$

$$\Rightarrow N = mg\cos\theta - 2mg(1 - \cos\theta) = 3mg\cos\theta - 2mg = 0$$

$$\Rightarrow \boxed{\cos\theta = \frac{2}{3}}$$

N

θ

mg

Problem 4: Tabletop Rocket [25 pts]

$$\gamma = \left| \frac{dM}{dt} \right|$$

$M_0/2$ $M_0/2$

$\vec{g}$

μ

v_{ex}

A rocket of total mass M_0, half of which is fuel, starts at rest on a long horizontal table. The coefficient of friction between the rocket and table surfaces is μ. At time t = 0, the rocket is ignited, ejecting fuel out at a constant rate $\gamma = |dM/dt|$ with velocity v_{ex} relative to the rocket. Constant gravitational acceleration g acts downward.

(a) [10 pts] What condition must be met for the rocket to start moving at t = 0?

(b) [10 pts] Assuming that the rocket satisfies this requirement, what is the maximum speed V_{MAX} achieved by the rocket?

(c) [5 pts] How far does the rocket go after it runs out of fuel? You can express your answer in terms of V_{MAX}.

(d) [BONUS 5 pts] How far does the rocket travel in total? For this you will need to make use of the following integral:

$$\int \ln u \, du = u \ln u - u$$

[SOLUTION]

(a) The condition is that the rocket thrust ($\gamma\mathbf{v}_{ex}$) must exceed the force due to friction ($\mu M_0 g$). Hence

$$\gamma v_{ex} > \mu M_0 g$$

(b) The maximum speed occurs when the rocket has used up all its fuel (note: since the friction force is not velocity-dependent, there is no "terminal velocity" in this case). Applying conservation of momentum including the impulse from friction over a time dt (see figure):
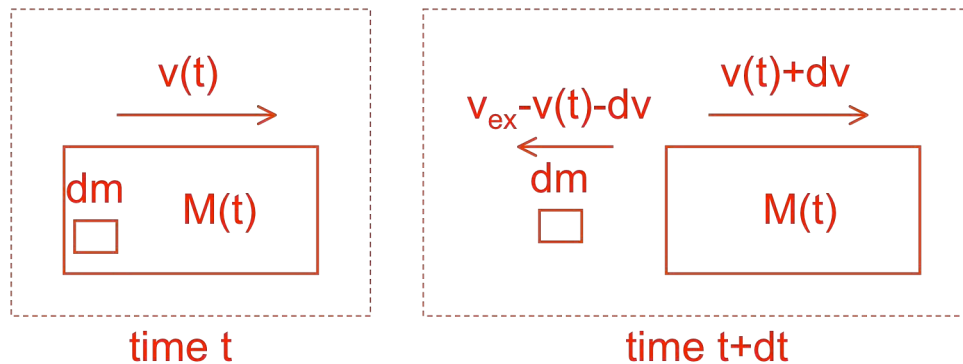
$$p_i = (M(t) + dm)v(t)$$

$$p_f = M(t)(v(t) + dv) - dm(v_{ex} - v(t) - dv)$$

$$p_f - p_i = M(t)dv - dmv_{ex} = F_f dt = -\mu g M(t) dt$$

$$\Rightarrow dv = v_{ex}\frac{dm}{M(t)} - \mu g dt = -v_{ex}\frac{dM}{M(t)} - \mu g dt$$

where we use the fact that a positive dm mass ejected corresponds to –dM lost from the rocket's total mass. The three terms can be separately integrated:

$$\int_o^{V_{max}} dv = -v_{ex}\int_{M_0}^{M_0/2}\frac{dM}{M} - \int_0^{t_{max}}\mu g dt$$

$$\Rightarrow V_{max} = -v_{ex} \ln\left[\frac{M_0}{2}\frac{1}{M_0}\right] - \mu g t_{max}$$

t_{max} is the time it takes for the fuel to run out, which is $(M_0/2)/\gamma$. Hence,

$$V_{max} = v_{ex}\ln 2 - \mu g \frac{M_0}{2\gamma}$$

(c) After the rocket runs out of fuel, the problem reduces to a simple mass accelerated by a constant friction force:

$$\frac{M_0}{2}\ddot{x} = -\mu\frac{M_0}{2}g$$

We can also solve this using energy, since the work done on the rocket to reduce its kinetic energy to 0 is simply the friction force acting over the distance the rocket travels:

$$\Delta K = \frac{1}{2}\frac{M_0}{2}V_{max}^2 = \mu\frac{M_0}{2}g\Delta x$$

$$\Rightarrow \Delta x = \frac{V_{max}^2}{2\mu g}$$

(d) For those who attempted it, we simply use the expression of velocity as a function of time from our derivation of the rocket equation above, and integrate this to get the distance traveled:

$$v(t) = \frac{dx}{dt} = -v_{ex}\ln\left[\frac{M(t)}{M_0}\right] - \mu g t$$

$$\Rightarrow dx = -v_{ex}\ln\left[\frac{M_0 - \gamma t}{M_0}\right]dt - \mu g t dt$$

where we now explicitly put in the time dependence of the rocket mass. To integrate the first term on the right, we make the substitution:
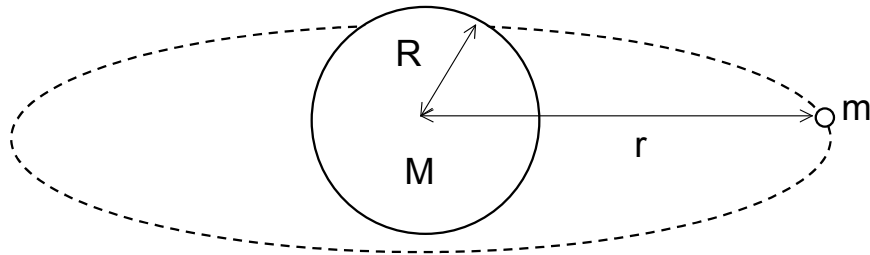
$$u = 1 - \frac{\gamma}{M_0} t \Rightarrow du = -\frac{\gamma}{M_0} dt$$

Using our substitution for t_{max} for the limits of u and solving the left side and second term on the right side:

$$\Delta x_f = \frac{v_{ex} M_0}{\gamma} \int_1^{1/2} \ln u \, du - \frac{1}{2} \mu g t_{max}^2$$

$$= \frac{v_{ex} M_0}{\gamma} [u \ln u - u]_1^{1/2} - \frac{1}{8} \mu g \frac{M_0^2}{\gamma^2}$$

$$= \frac{v_{ex} M_0}{2\gamma} (1 - \ln 2) - \frac{1}{8} \mu g \frac{M_0^2}{\gamma^2}$$

The total distance traveled is then

$$\Delta x_{total} = \Delta x_f + \Delta x = \boxed{\frac{v_{ex} M_0}{2\gamma} (1 - \ln 2) + \frac{V_{max}^2}{2\mu g} - \frac{1}{8} \mu g \frac{M_0^2}{\gamma^2}}$$

Problem 5: Planet Orbit [25 pts]

A small planet of mass m is in a circular orbit of radius r around a star of mass M and radius R in otherwise empty space (assume M >> m so the star is stationary).

(a) [10 pts] Determine the potential energy U(r), the kinetic energy K(r) and the total mechanical energy E(r) of the planet in terms of G, M and r assuming U→0 as r→∞.

(b) [5 pts] Determine the minimum amount of mechanical energy that must be added to the planet to cause it to escape from the star (i.e., r→∞). By what <u>factor</u> must the speed of the planet be increased to cause it to escape?

(c) [5 pts] Now assume that the planet in subject to a viscous force of the form

$$\vec{F} = -Amv^2\hat{v}$$

where A is a constant and $\hat{v}$ is the direction of motion. Compute the loss of mechanical energy in one orbital period in terms of G, M, r and A. Assume that this loss is small enough that neither the orbital radius nor speed of the planet changes appreciably in one orbit.

(d) [5 pts] Building from (c), compute the change in radius of the planet in one orbital period due to the viscous force and the corresponding radial velocity based on the assumptions above, in terms of G, M, r and A. Does the planet fall into the star or away from it?

[SOLUTIONS]

(a) The potential can be derived from the force law:

$$U(r) = -\int \vec{F} \cdot d\vec{r} = \int \frac{GMm}{r^2} dr = -\frac{GMm}{r} + C$$

In this case we know that the constant $C = 0$ since $U(\infty) = 0$. Hence:

$$\boxed{U(r) = -\frac{GMm}{r}}$$

Note the m is a factor in this and other expressions (mistakenly neglected in the question).

The kinetic energy is

$$K(r) = \frac{1}{2}mv^2 = \frac{1}{2}mr^2\dot{\theta}^2$$

but we can substitute this using the force law in polar coordinates:

$$m(\ddot{r} - r\dot{\theta}^2) = -\frac{mv^2}{r} = -\frac{GMm}{r^2}$$

$$\Rightarrow \boxed{K(r) = \frac{GMm}{2r}}$$

the total mechanical energy E = K + U is then:

$$\boxed{E(r) = -\frac{GMm}{2r}}$$

(b) The work required to effectively bring the planet to r = ∞ is simply that needed to climb the potential well to U = 0. In other words:

$$\Delta K = -U(r) = \boxed{\frac{GMm}{2r}}$$

So the kinetic energy must double, which means the speed must increase by a factor of $\boxed{\sqrt{2}}$.

(c) The loss of mechanical energy simply comes from the work done by the frictional force over one orbit (note: this is formally true since the potential energy doesn't change – work only acts to change kinetic energy). Hence:

$$\Delta E = \Delta K = \int \vec{F} \cdot d\vec{r} = -2\pi r A m v^2 = \boxed{-2\pi A G M m}$$

where the substitution for mv² from above is used

(d) We can also write the change in energy as:

$$\Delta E = \Delta \left(-\frac{GMm}{2r} \right) = \frac{GMm}{2r^2} \Delta r = -2\pi A G M m$$

$$\Rightarrow \boxed{\Delta r = -4\pi A r^2}$$

to compute a velocity, we note that this change in r occurs over one orbit period for which:

$$\Delta t = \frac{2\pi r}{v} = 2\pi r \sqrt{\frac{r}{GM}}$$

Thus:

$$\boxed{\frac{\Delta r}{\Delta t} \approx \frac{dr}{dt} = -2A\sqrt{GMr}}$$

Note that the radial velocity is negative, so the planet falls in to the star.