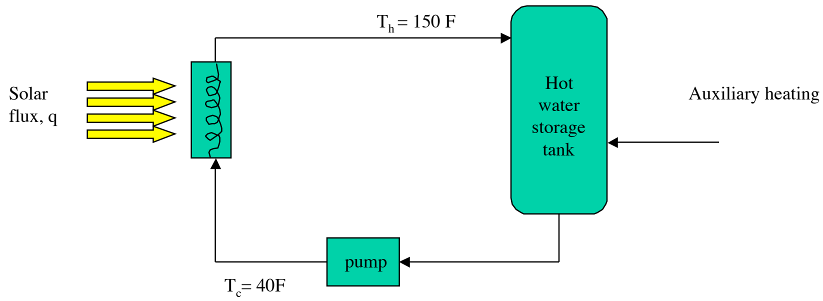
The Time Value of Money

February 9, 2004

Solar Hot Water Heating System

- Requirement: 100 gpd @ 150°F



BTU required/day = 100 (gpd) x 8.33 (lb/gal) x 1 (BTU/lb-°F) x (150-40)(°F)

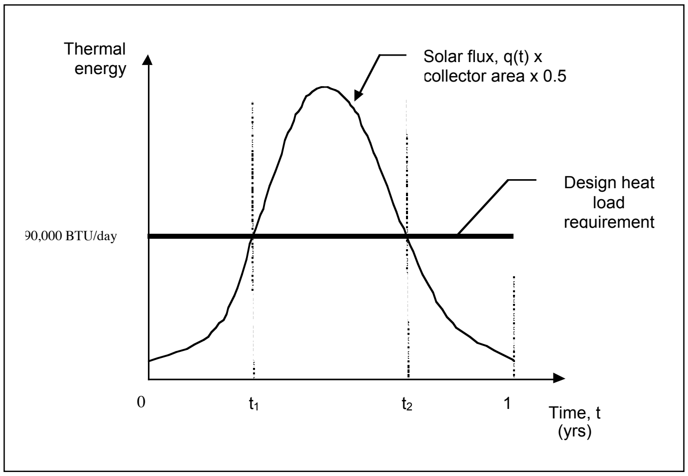
= ~ 90,000 BTU/day

Solar Hot Water Heating System (contd.)

Solar BTU delivered per sq. ft. per day	January	June	Average
Boston,MA	500	2000	1000
Tucson, AZ	1000	2500	2000

- Assume: We use the solar heater to provide 50% of annual heat load (with the remainder being supplied by auxiliary heater).
- So, if the collector efficiency is 50%, we need:
  - 90 sq. ft. of collector area in Boston
  - 45 sq. ft. of collector area in Tucson

  and the collectors will deliver 45,000 BTU/day, on average.

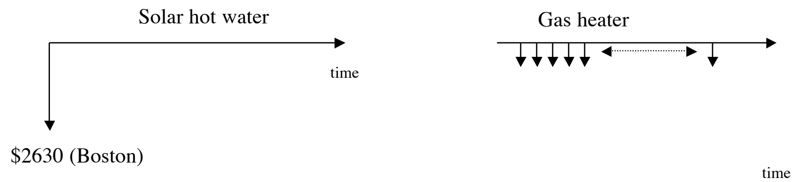


General problem: Choose collector area such that capital cost of collector + cost of auxiliary energy is minimized.

System Cost	Boston @ 90 sq. ft	Tucson @ 45 sq ft
Panels ($17/sq.ft)	1530	765
Piping	500	500
Pump & controls	100	100
Installation	500	500
Total	$2630	$1865

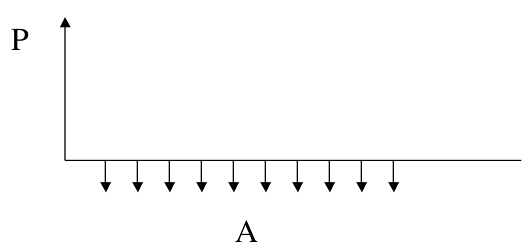
Note: We don't include the cost of the hot water storage tank, since the gas heater also requires such a tank.

Economic Comparison



- A simplistic comparison: Assume equipment lifetime of 10 years. Hence, "annual cost" = $263. Avoided cost of natural gas = 45,000 BTU/day x 10^{-6} (MCF/BTU) x 14 ($/MCF) x 365 days/yr x (1/0.8) = $287/yr
- But this calculation understates the true cost of the solar investment
- Suppose the homeowner borrows the funds for the solar heating system.
  - Assume the loan is for 10 years, at 10%/yr interest rate, and that the bank requires 10 <u>equal</u> annual payments.
  - If the homeowner simply paid the bank $263/yr for 10 years, the bank wouldn't be happy!

Economic comparison (contd.)



- What is the value of A, the uniform annual payment, such that the loan P (at interest rate r) will be fully repaid after 10 years, with interest? (Assume payments are made at the end of each year.)

After the end of the first year, the homeowner pays interest at rate r on the principal, rP, and retires a portion of the principal, D_1, where

$$A = Pr + D_1 \qquad (1)$$

After the end of the second year, the homeowner pays interest on the residual principal of $(P-D_1)$ and retires a further portion of the principal, D_2, where

$$A = (P-D_1)r + D_2 \qquad (2)$$

And substituting for D_1 in (2) and solving for D_2 we have:

$$D_2 = (A - Pr)(1+r) \qquad (3)$$

After the end of the third year, the homeowner pays interest on the residual principal, $P - D_1 - D_2$, and retires a further portion of the principal D_3, where

$$A = (P-D_1-D_2)r + D_3 \qquad (4)$$

And substituting for D_1 and D_2 in (4) and solving for D_3, we have:

$$D_3 = (A - Pr)(1+r)^2 \qquad (5)$$

And, by induction,

$$D_n = (A - Pr)(1 + r)^{n-1} \qquad (6)$$

And since

$$\sum_{n=1}^{N} D_n = P$$

we can write

$$P = (A - Pr)(1 + (1 + r) + (1 + r)^2 + \ldots + (1 + r)^{N-1})$$
$$= (A - Pr)\left[\frac{1 - (1 + r)^N}{-r}\right]$$

and solving for A, we have

$$\boxed{A = P\left[\frac{r(1+r)^N}{(1+r)^N - 1}\right] \qquad (1)}$$

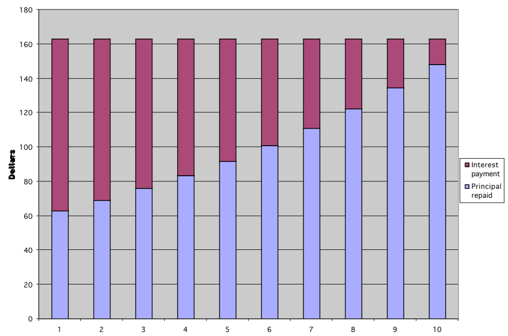
Loan payment schedule

Constant annual payments

Loan duration: 10 yrs

Interest rate: 10%/yr

$$\boxed{A = P\left[\frac{r(1+r)^N}{(1+r)^N - 1}\right]}$$



What is the price of natural gas above which the solar heating system will be economic?

We solve for p^* in:

$$A = I_o\left[\frac{r(1+r)^N}{(1+r)^N - 1}\right] = p^* Q$$

where:

I_o = investment cost of solar hot water heating system

P^*= breakeven price of natural gas ($/thousand cu. ft., ($/MCF))

Q = annual gas requirement (in MCF)

= 45,000 (BTU/day) x 365 (day/yr) x 10^{-6} (MCF/BTU) x 1/0.8

= <u>20.5 MCF/yr</u>

(where we have again assumed an 80% efficiency for the gas heater)

Minimum delivered price of natural gas above which residential solar hot water heating is economical ($/MCF)

10-yr loan w/ uniform annual payments @ interest rate r (%/yr)	Threshold price of gas, p* ($/MCF)	
	Boston (I_o = $2630)	Tucson (I_o = $1865)
3	15	10.5
6	17.4	12.5
10	20.9	14.8

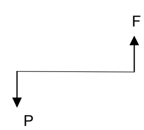
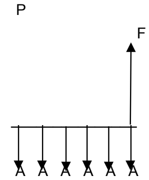
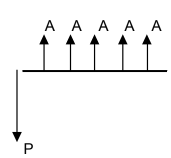
Note:

Average price of residential natural gas in Massachusetts during 2002 = ~$15/MCF

Average price of residential natural gas in Arizona during 2002 = $12.36/MCF

(Source: DOE Energy Information Administration web site)

Time Value Equivalence Factors (Discrete compounding, discrete payments)

Factor Name	Factor Notation	Formula	Cash Flow Diagram
Future worth factor (compound amount factor)	(F/P, i, N)	$F=P(1+i)^N$	
Present worth factor	(P/F, i, N)	$P=F(1+i)^{-N}$	
Uniform series compound amount factor (aka future-worth-of-an-annuity factor)	(F/A, i, N)	$F = A\left[\frac{(1+i)^N - 1}{i}\right]$	
Sinking fund factor	(A/F, i, N)	$A = F\left[\frac{i}{(1+i)^N - 1}\right]$	
Present worth of an annuity factor	(P/A, i, N)	$P = A\left[\frac{(1+i)^N - 1}{i(1+i)^N}\right]$	
Capital recovery factor	(A/P, i, N)	$A = P\left[\frac{i(1+i)^N}{(1+i)^N - 1}\right]$	

Frequency of compounding

- Compounding of interest occurs over different intervals -- annually, quarterly, monthly, daily, etc.
- Example: A loan offered at "an annual interest rate of 12%, compounded quarterly":
  - Interest rate per quarter = 12/4 = 3%
  - Effective annual interest rate = $(1+0.03)^4 - 1 = 1.1255-1 = 0.1255$ (i.e., 12.55%)
- Differentiate between <u>nominal</u> and <u>effective</u> interest rates:
  - If i is the interest rate per period, and m is the number of compounding periods per year:
    - Effective annual interest rate, $i_a = (1+i)^m - 1$
    - Nominal interest rate, $r = i.m$
    - And, $i_a = (1 + r/m)^m - 1$

Effective interest rates, i_a, for various nominal rates, r, and compounding frequencies,m

Compounding frequency	Compounding periods per year,m	Effective rate i_a for nominal rate of					
		6%	8%	10%	12%	15%	24%
Annually	1	6.00	8.00	10.00	12.00	15.00	24.00
Semiannually	2	6.09	8.16	10.25	12.36	15.56	25.44
Quarterly	4	6.14	8.24	10.38	12.55	15.87	26.25
Bimonthly	6	6.15	8.27	10.43	12.62	15.97	26.53
Monthly	12	6.17	8.30	10.47	12.68	16.08	26.82
Daily	365	6.18	8.33	10.52	12.75	16.18	27.11
Continuous	∞	6.18	8.33	10.52	12.75	16.18	27.12

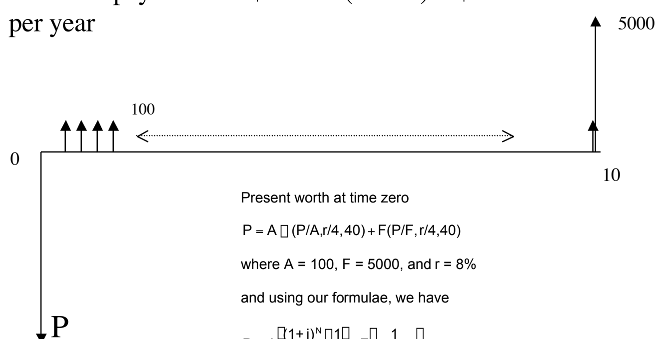
Example -- Valuation of Bonds

- Bonds are sold by organizations to raise money
- The bond represents a debt that the organization owes to the bondholder (not a share of ownership)
- Bonds typically bear interest semi-annually or quarterly, and are redeemable for a specified maturity value (also known as the face value) at a given maturity date.
- Interest is paid in the form of regular 'premiums'. The flow of premiums constitutes an annuity, A, where

  A = (face value) x (bond rate)
- The value of a bond at a given point in time is equal to the present worth of the remaining premium payments plus the present worth of the redemption payment (i.e., the face value)

Example -- Valuation of Bonds (contd.)

- Consider a 10-year U.S. treasury bond with a face value of \$5000 and a bond rate of 8 percent, payable quarterly:
  - Premium payments of \$5000 x (0.08/4) = \$100 occur four times per year



Present worth at time zero

$P = A(P/A, r/4, 40) + F(P/F, r/4, 40)$

where A = 100, F = 5000, and r = 8%

and using our formulae, we have

$$P = A\left[\frac{(1+i)^N - 1}{i(1+i)^N}\right] + F\left[\frac{1}{(1+i)^N}\right]$$

and since

$A = Fi$

$P = F$

Continuous compounding

- For the case of m compounding periods per year and nominal annual interest rate, r, the effective annual interest rate i_a is given by:

  $i_a = (1 + r/m)^m - 1$
- In the limiting case of continuous compounding

$$i_a = \lim_{m \to \infty} \left(1 + \frac{r}{m}\right)^m - 1$$

Writing

$$i = r/m$$

$$i_a = \lim_{i \to 0} (1+i)^{\frac{r}{i}} - 1$$
$$= e^r - 1$$

or

$$r = \ln(1 + i_a)$$

Effective interest rates, i_a, for various nominal rates, r, and compounding frequencies, m

Compounding frequency	Compounding periods per year,m	Effective rate i_a for nominal rate of 6%	8%	10%	12%	15%	24%
Annually	1	6.00	8.00	10.00	12.00	15.00	24.00
Semiannually	2	6.09	8.16	10.25	12.36	15.56	25.44
Quarterly	4	6.14	8.24	10.38	12.55	15.87	26.25
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Daily	365	6.18	8.33	10.52	12.75	16.18	27.11
Continuous	**∞**	**6.18**	**8.33**	**10.52**	**12.75**	**16.18**	**27.12**

Continuous Compounding, Discrete Cash Flows

(nominal annual interest rate r, continuously compounded, N periods)

To Find	Given	Factor Name	Factor Symbol	Factor formula
F	P	Future Worth Factor*	(F/P, r%, N)	F = $P(e^{rN})$
P	F	Present Worth Factor	(P/F, r%, N)	$P = F(e^{-rN})$
F	A	Future Worth of an annuity factor	(F/A, r%, N)	$F = A\left[\frac{e^{rN}-1}{e^{r}-1}\right]$
A	F	Sinking Fund Factor	(A/F, r%, N)	$A = P\left[\frac{e^{r}-1}{e^{rN}-1}\right]$
P	A	Present Worth of an annuity Factor	(P/A, r%, N)	$P = A\left[\frac{e^{rN}-1}{e^{rN}(e^{r}-1)}\right]$
A	P	Capital Recovery Factor	(A/P, r%, N)	$A = P\left[\frac{e^{rN}(e^{r}-1)}{e^{rN}-1}\right]$

Example:

You need $25,000 immediately in order to make a down payment on a new home. Suppose that you can borrow the money from your insurance company. You will be required to repay the loan in equal payments, made every 6 months over the next 8 years. The nominal interest rate being charged is 7% compounded continuously. What is the amount of each payment?

Example -- payment and compounding periods don't coincide

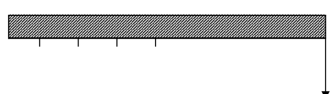
Find the present worth of a series of quarterly payments of \$1000 extending over 5 years, if the nominal interest rate is 8%, compounded monthly.

Continuous cash flows, continuous compounding

In many applications, cash flows are also essentially continuous (or it is convenient to treat them as such). We need to develop time value factors equivalent to those we have obtained for discrete cash flows.

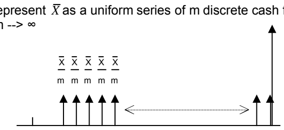
Let $\overline{X}$ = continuous rate of flow of cash over a period (in units of , e.g., \$/yr)

Assume also: continuous compounding at a nominal rate of r%/yr.



First, derive a future (end of year) worth X equivalent to 1 year of continuous cash flow $\overline{X}$ (\$/yr).

To do this, represent $\overline{X}$ as a uniform series of m discrete cash flows of magnitude $\overline{X}$/m in the limit as m --> ∞



$$FW = \lim_{m \to \infty} \frac{\overline{X}}{m} \cdot (F/A,\ i\%,\ m) \quad \text{where} \quad i = \frac{r}{m}$$

$$X = \lim_{m \to \infty} \frac{\overline{X}}{m} \left[\frac{(1+i)^m - 1}{i} \right] = \lim_{i \to 0} \overline{X} \left[\frac{(1+i)^{r/i} - 1}{r} \right]$$

$$= \overline{X} \left[\frac{e^r - 1}{r} \right] = \overline{X} \frac{i_a}{\ln(1+i_a)}$$

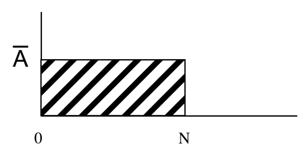
<u>Example of continuous cash flows & continuous compounding</u>

An oil refinery is considering an investment in upgrading a main pump. The upgrading is expected to result in a reduction in maintenance labor and materials costs of about $3000 per year.

If the expected lifetime of the pump is three years, what is the largest investment in the project that would be justified by the expected savings?

(Assume that the required rate of return on investments (before taxes) is a nominal rate of 20%/ year, continuously compounded.)

"Funds flow" time value factors for continuous compounding



To Find:	Given:			
P	$\overline{A}$	$(P/\overline{A}, r\%, N)$	Present worth of a continuous annuity factor	$\dfrac{e^{rN}-1}{re^{rN}}$
$\overline{A}$	P	$(\overline{A}/P, r\%, N)$	Continuous annuity from a present amount factor	$\dfrac{re^{rN}}{e^{rN}-1}$
F	$\overline{P}$	$(F/\overline{P}, r\%, N)$	Future worth of a continuous present amount factor	$\dfrac{e^{rN}(e^{r}-1)}{re^{r}}$
P	$\overline{F}$	$(P/\overline{F}, r\%, N)$	Present worth of a continuous future amount factor	$\dfrac{e^{r}-1}{re^{rN}}$

Example (from PSB):

A county government is considering building a road from downtown to the airport to relieve congested traffic on the existing two-lane divided highway. Before allowing the sale of a bond to finance the road project, the county court has requested an estimate of future toll revenues over the bond life. The toll revenues are directly proportional to the growth of traffic over the years, so the following growth cash flow function is assumed to be reasonable:

$$F(t) = 5\,(1 - e^{-0.1t}) \qquad \text{(in millions of dollars)}$$

The bond is to be a 25-year instrument, and will pay interest at an annual rate of 6%, continuously compounded.

The Method of Laplace Transforms

Note the similarity between the expression for finding the present value of a continuous cash flow stream, f(t) and the Laplace transform of the function

$$P = \int_0^N f(t)e^{-rt}dt$$

and

$$\mathcal{L}\{\mathrm{f(t)}\} = \int_0 e^{-st} f(t)dt$$

Tables of Laplace transforms for a range of functions are available in many mathematical handbooks, and the Laplace Transform method can be used as an alternative to the traditional approach to evaluate equivalence values for continuous cash flows. For simple cash flow problems, there is not much computational advantage in using this method, but for complex cash flow situations the Laplace transform method may offer significant savings in computation.

For more details of the Laplace transform technique for cash flow modeling, see Park and Sharp-Bette, Chapter 3.

Example

Find the present worth of a seasonal cash flow given by:

$$f(t) = \overline{A} \sin^2 \square t$$

where f(t) is given in \$/yr, t is in years, and the cash flow is discounted at an annual rate r, continuously compounded