22.616. Problem Set #6 Solutions

1. Ambipolar Potential in a Magnetized plasma Column

The radial current accounts for viscousity and inertial rotation

$$J_r = \frac{c}{B}\left(-\frac{1}{r}\frac{\partial}{\partial r} r n_i m_i \eta_i \frac{\partial}{\partial r} V_{i\theta} + n_i m_i \frac{\partial}{\partial t} V_{i\theta}\right)$$

By ambipolarity. intrinsically, $J_r = 0$. i.e.

$$\frac{\partial}{\partial t} V_{i\theta} = \eta_i \frac{1}{r}\frac{\partial}{\partial r} r \frac{\partial}{\partial r} V_{i\theta}$$

it is a diffusion equation in cylindrical system.

A general solutions is

$$V_{i\theta} = \sum_n A_n e^{-K_n^2 t} J_0\left(\frac{K_n}{\sqrt{\eta_i}} r\right)$$

when $t \to \infty$, Asymptotically, $V_{i\theta} = 0$.

Usually (main term)

$$V_{i\theta} = \frac{c\, E \times B}{B^2} + \frac{b \times \nabla P_i}{n_i m_i \Omega_i}$$

$$= -\frac{cE_r}{B} + \frac{1}{n_i m_i \Omega_i}\frac{dP_i}{dr} = 0$$

So $$E_r = \frac{B}{n_i m_i \Omega_i c}\frac{dP_i}{dr}$$. i.e.

$$-\frac{d\phi}{dr} = \frac{1}{n_i e_i}\frac{dP_i}{dr} = \frac{1}{e_i}\frac{dT_i}{dr}$$

$\Rightarrow \quad e_i\phi(r) = \cancel{n_i} T_i(r)$, (Assume at edge $e\phi(a) = T_i(a) = 0$)

Therefore in this steady ~~steat~~ state $V_{i\theta} = 0$. the net rotation ~~is only from electrons~~ will die away. The electron velocity ~~rotation~~

$$V_{e\theta} = \frac{c\,E\times B}{B} + \frac{b\times\nabla P_e}{n_e m_e \Omega_e}$$ will contribute to current Not rotation.

Note the friction induced radial electron and ion flow cancel each other. to keep the ambipolarity.

2. self-Adjoint property of Collision operator

$$C(\hat{f}) = \frac{\Gamma}{2}\frac{\partial}{\partial \underline{v}}\cdot\int d^3v' f_0 f_0' \,\underline{\underline{U}}\cdot\left(\frac{\partial}{\partial \underline{v}}\hat{f} - \frac{\partial}{\partial \underline{v}'}\hat{f}'\right)$$

$$\int d^3v\, \hat{g}\, C(\hat{f}) = \frac{\Gamma}{2}\int d^3v\, \hat{g}\,\frac{\partial}{\partial \underline{v}}\cdot\int d^3v' f_0 f_0' \,\underline{\underline{U}}\cdot\left(\frac{\partial}{\partial \underline{v}}\hat{f} - \frac{\partial \hat{f}'}{\partial \underline{v}'}\right)$$

Integrate by parts (Assume $\hat{f}$ & $\hat{g}$ is reasonably behaved at ∞)

$$\int d^3v\, \hat{g}\, C(\hat{f}) = -\frac{\Gamma}{2}\int d^3v\, d^3v' f_0 f_0' \frac{\partial \hat{g}}{\partial \underline{v}}\cdot\underline{\underline{U}}\cdot\left(\frac{\partial}{\partial \underline{v}}\hat{f} - \frac{\partial \hat{f}'}{\partial \underline{v}'}\right) \quad -(1)$$

make change $\underline{v} \rightleftharpoons \underline{v}'$, $\underline{\underline{U}}$ is symmetric about $\underline{v}$ and $\underline{v}'$

$$\int d^3v\, \hat{g}\, C(\hat{f}) = -\frac{\Gamma}{2}\int d^3v\, d^3v' f_0 f_0' \left(-\frac{\partial \hat{g}'}{\partial \underline{v}'}\right)\cdot\underline{\underline{U}}\cdot\left(\frac{\partial}{\partial \underline{v}}\hat{f} - \frac{\partial}{\partial \underline{v}'}\hat{f}'\right) \quad -(2)$$

$\frac{① + ②}{2} \Rightarrow$

$$\int d^3v\, \hat{g}\, C_e(\hat{f}) = -\frac{\Gamma}{4}\int d^3v\, d^3v'\, f_0 f_0' \left(\frac{\partial \hat{g}}{\partial \underline{v}} - \frac{\partial \hat{g}'}{\partial \underline{v}'}\right) \cdot \underline{\underline{U}} \cdot \left(\frac{\partial}{\partial \underline{v}}\hat{f} - \frac{\partial}{\partial \underline{v}'}\hat{f}'\right)$$

~~this~~ is a symmetric form about $\hat{g}$ & $\hat{f}$, so.

$$\int d^3v\, \hat{g}\, C_e(\hat{f}) = \int d^3v\, \hat{f}\, C_e(\hat{g}) \quad \text{, (self-adjoint !)}$$

3. Conservation of laws for linearized Collision operator

Use the self-adjointness of C_e.

$$\int d^3v \begin{bmatrix} 1 \\ m\underline{v} \\ \frac{1}{2}mv^2 \end{bmatrix} C_e(\hat{f}) = \int d^3v\, \hat{f}\, C_e\left(\begin{bmatrix} 1 \\ m\underline{v} \\ \frac{1}{2}mv^2 \end{bmatrix}\right)$$

$$C_e\left(\begin{bmatrix} 1 \\ m\underline{v} \\ \frac{1}{2}mv^2 \end{bmatrix}\right) = \frac{\Gamma}{2}\frac{\partial}{\partial \underline{v}} \cdot \int d^3v'\, f_0 f_0'\, \underline{\underline{U}} \cdot \left(\frac{\partial}{\partial \underline{v}}\begin{bmatrix} 1 \\ m\underline{v} \\ \frac{1}{2}mv^2 \end{bmatrix} - \frac{\partial}{\partial \underline{v}'}\begin{bmatrix} 1 \\ m\underline{v}' \\ \frac{1}{2}mv'^2 \end{bmatrix}\right)$$

~~be~~.

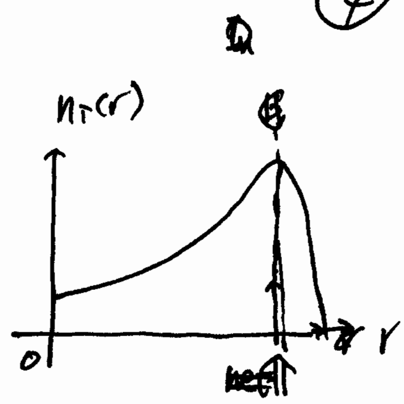
$$\frac{\partial}{\partial \underline{v}} 1 - \frac{\partial}{\partial \underline{v}'} 1 = 0$$

$$\frac{\partial}{\partial \underline{v}} \underline{v} - \frac{\partial}{\partial \underline{v}'} \underline{v}' = \underline{\underline{I}} - \underline{\underline{I}} = 0$$

$$\frac{\partial}{\partial \underline{v}} \frac{1}{2}mv^2 - \frac{\partial}{\partial \underline{v}'} \frac{1}{2}mv'^2 = m(\underline{v} - \underline{v}') \text{, but } \underline{\underline{U}} \cdot (\underline{v} - \underline{v}') = 0$$

So $$\int d^3v\, \hat{f}\, C_e\left(\begin{bmatrix} 1 \\ m\underline{v} \\ \frac{1}{2}mv^2 \end{bmatrix}\right) = 0 \Rightarrow \int d^3v \begin{bmatrix} 1 \\ m\underline{v} \\ \frac{1}{2}mv^2 \end{bmatrix} C_e(\hat{f}) = 0$$

4. Solution:



Inject neutral source

The radial ion flux is

$$\Gamma_i = -n_i \frac{T_i}{m_i \Omega_i^2 \tau_{iz}} \left(\frac{p_i'}{p_i} - \frac{T_z}{Z T_i} \frac{p_z'}{p_z} - \frac{3}{2} \frac{T_i'}{T_i} \right)$$

From the ambipolarity condition

$$\sum_a e_a \Gamma_a = e(\Gamma_i + Z\Gamma_z - \Gamma_e)$$

But $\Gamma_e \propto n_e \frac{T_e}{m_e \Omega_e \tau_e} \ll \Gamma_i$, so $\Gamma_z = -\frac{1}{Z}\Gamma_i$.

This is to say, if we can make ions flow in, the impurities would flow out.

If we take ~~some~~ $\frac{p_i'}{p_i} - \frac{T_z}{Z T_i}\frac{p_z'}{p_z} - \frac{3}{2}\frac{T_i'}{T_i} > 0$, ion will flow inward.

Assume $T_z \sim T_i$, then $\left(\ln n_i - \frac{1}{Z}\ln n_z\right)' > \left(\ln T_i \left(\frac{1}{2} + \frac{1}{Z}\right)\right)'$

~~i.e. If $Z \gg 1$~~ :
$$\frac{n_i(r)}{n_i(0)} > \left(\frac{n_z(r)}{n_z(0)}\right)^{\frac{1}{Z}} \left(\frac{T_i(r)}{T_i(0)}\right)^{\frac{1}{2}-\frac{1}{Z}} \quad \text{---} \langle 1 \rangle$$

So if we can add some ~~net~~ neutral source at the tokamak edge to make a density (pressure) profile to satisfy Condition $\langle 1 \rangle$.

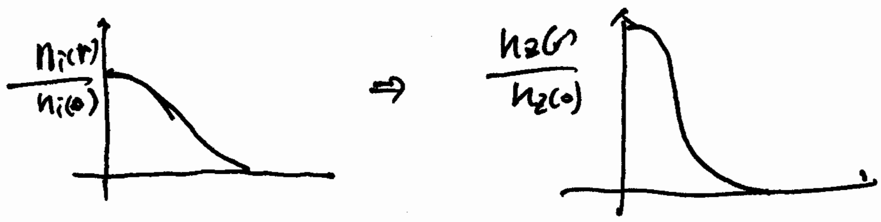
The ~~particle~~ ions will flow in until ~~re~~ a steady state is reached.

$\Gamma_i = 0$.
$$\frac{n_i(r)}{n_i(0)} = \left(\frac{n_z(r)}{n_z(0)}\right)^{\frac{1}{Z}} \left(\frac{T_i(r)}{T_i(0)}\right)^{\frac{1}{2}-\frac{1}{Z}}$$

So For Helium ash, $z=2$

$$\frac{n_i(r)}{n_i(0)} = \left(\frac{n_z(r)}{n_z(0)}\right)^{\frac{1}{2}} \Rightarrow \frac{n_z(r)}{n_z(0)} = \left(\frac{n_i(r)}{n_i(0)}\right)^2$$

Assume the steady ~~stead~~ state Ion distribution is



So there is a big accumulation of ~~ions~~ impurities (He) ~~at~~ in the center.

However, for high z impurities, e.g. Carbon, $Z=6$

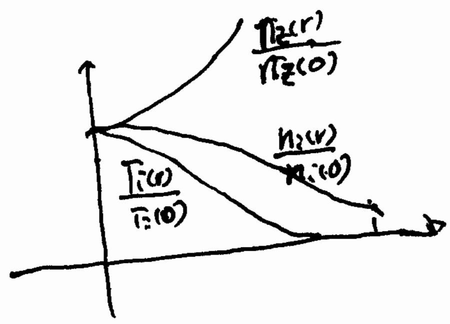
$$\frac{n_z(r)}{n_z(0)} = \left(\frac{n_i(r)}{n_i(0)}\right)^6 \left(\frac{T_i(r)}{T_i(0)}\right)^{-2}$$

There is a good chance that we can make $n_z(a) > n_z(0)$

For example. we can take $\frac{n_i(r)}{n_i(0)} = \frac{1}{1+4\frac{r^2}{a^2}}$

But if we ~~control the~~ make up a temperature profile, s.t. $\frac{T_i(r)}{T_i(0)} = \frac{1}{(1+4r^2/a^2)^4}$

~~There~~ Then $\frac{n_z(r)}{n_z(0)} = \left(1+\frac{4r^2}{a^2}\right)^2$, there would be no accumulation of impurities in the center.



5. Dimagnetic Fluxes.

$$n m V_{*y} = n\int d^3v\, m v_y f_1 = \frac{\cancel{e}m}{\Omega}\int d^3v\, v_y^2 \left(A_1 + A_2\left(\frac{mv^2}{2T} - \frac{5}{2}\right)\right) f_M$$

1° $$\int d^3v\, v_y^2 f_M = \frac{1}{3}\int d^3v\, v^2 \frac{n}{(2\pi T/m)^{3/2}} e^{-\frac{mv^2}{2T}}$$

$$= \frac{n}{3}\left(\frac{m}{2\pi T}\right)^{3/2} \times 4\pi \int_0^\infty dv\, v^4 e^{-\frac{mv^2}{2T}}$$

let $x = \sqrt{\frac{m}{2T}}\, v$, then

$$\int d^3v\, v_y^2 f_M = \frac{4\pi n}{3}\left(\frac{m}{2\pi T}\right)^{3/2}\left(\frac{2T}{m}\right)^{5/2}\underbrace{\int_0^\infty dx\, x^4 e^{-x^2}}_{\frac{3\sqrt{\pi}}{8}}$$

$$= \frac{nT}{m}$$

2° $$\int d^3v\, v_y^2 \left(\frac{mv^2}{2T} - \frac{5}{2}\right) f_M$$

$$= \frac{1}{3}\int d^3v\, v^2 \left(\frac{mv^2}{2T} - \frac{5}{2}\right) f_M$$

$$= \frac{4\pi n}{3}\left(\frac{m}{2\pi T}\right)^{3/2}\int_0^\infty dv\, v^4 \left(\frac{mv^2}{2T} - \frac{5}{2}\right) e^{-\frac{mv^2}{2T}}$$

(let $x = \sqrt{\frac{m}{2T}}\, v$, then)

$$= \frac{4\pi n}{3}\left(\frac{m}{2\pi T}\right)^{3/2}\left(\frac{2T}{m}\right)^{2}\int_0^\infty dx\, x^4 \left(x^2 - \frac{5}{2}\right) e^{-x^2}$$

~~But~~ let $t = x^2$, then

$$\int_0^\infty dx\, x^4 (x^2 - \tfrac{5}{2}) e^{-x^2}$$

$$= \frac{1}{2} \int_0^\infty dt\, t^{3/2} (t - \tfrac{5}{2}) e^{-t}$$

$$= \frac{1}{2} \left(\Gamma(\tfrac{7}{2}) - \tfrac{5}{2} \Gamma(\tfrac{5}{2}) \right)$$

$$= 0$$

So. we have $\int d^3v\, v_y^2 \left(\frac{mv^2}{2T} - \frac{5}{2} \right) f_m = 0$

Then $n m V_{ry} = \frac{m}{\Omega} \frac{nT}{m} A_1 = \frac{P}{\Omega} A_1 \propto A_1$

6. Generalized Flux-Friction Relations

Take the energy flux moment of the leading order kinetic equation

$$\Omega\, \underline{v} \times \underline{b} \cdot \frac{\partial}{\partial \underline{v}} f \simeq C(f, f)$$

$$\Rightarrow \int d^3v\, \frac{1}{2} m v^2 \underline{v}\, \Omega\, \underline{v} \times \underline{b} \cdot \frac{\partial}{\partial \underline{v}} f = \int d^3v\, \frac{1}{2} m v^2 \underline{v}\, C(f, f)$$

But $\left(\int d^3v\, \frac{1}{2} m v^2 \underline{v}\, \Omega\, \underline{v} \times \underline{b} \cdot \frac{\partial}{\partial \underline{v}} f \right)_i$

$$= \frac{\Omega m}{2} \int d^3v\, v^2 v_i\, \epsilon_{jkl} v_k b_l \frac{\partial f}{\partial v_j}$$

Integrate by parts

$$= -\frac{\Omega m}{2}\int d^3v\, f\, \varepsilon_{jkl} b_l \frac{\partial}{\partial v_j}(v^2 v_i v_k)$$

$$= -\frac{\Omega m}{2}\int d^3v\, f\, \varepsilon_{jkl} b_l (2v_j v_i v_k + v^2\delta_{ij} v_k + v^2 \delta_{jk} v_i)$$

$$(\varepsilon_{jkl}\delta_{jk} = 0,\quad \varepsilon_{jkl} b_l v_j v_k = v\cdot(v\times b) = 0)$$

$$= -\frac{\Omega m}{2}\int d^3v\, f\, \varepsilon_{ikl} b_l v^2 v_k$$

Therefore we can write above result in vector form

$$\int d^3v\, \frac{1}{2}mv^2 \underline{v}\, \Omega\, \underline{v}\times \underline{b}\cdot\frac{\partial}{\partial \underline{v}} f = \Omega\, \underline{b}\times\int d^3v\, \frac{1}{2}mv^2\underline{v}$$

$$= -\Omega\, \underline{Q}\times\underline{b} = \Omega\, Q_x \underline{e}_y - \Omega Q_y \underline{e}_x$$

$$\Rightarrow \quad \Omega Q_x = +\int d^3v\, \frac{1}{2}mv^2 v_y\, C_e(f,f)$$

$$= +\int d^3v\, \frac{1}{2}mv^2 v_y\, C_e(\hat{f})$$

$$(\quad f = (\hat{f}+1) f_0 \quad)$$

7. like-particle (Ion) Collision Fluxes

Assume the kinetic equation for the guiding center distribution

$$\frac{\partial f}{\partial t} = C_0(f,f) + C_2(f,f)$$

where C_0 is the "velocity" operator

$$C_0(f,f) \equiv \frac{\Gamma^{ii}}{2}\frac{\partial}{\partial \underline{v}}\cdot\int d^3v'\, \underline{\underline{U}}\cdot\left(\frac{\partial}{\partial \underline{v}} - \frac{\partial}{\partial \underline{v}'}\right) f f'$$

$$C_2(f,f) = \frac{\partial}{\partial \underline{X}}\int d^3v'\, D(\underline{v},\underline{v}')\left(\frac{\partial f}{\partial \underline{X}} f' - f\frac{\partial f'}{\partial \underline{X}}\right)$$

$$D(\underline{v},\underline{v}') \equiv \Gamma^{ii}\, \underline{e}_y\cdot\underline{\underline{U}}\cdot\underline{e}_y\, \frac{1}{\Omega^2}$$

Zeroth order

$$C_0(f_0,f_0) = 0 \quad \Rightarrow f_0 = f_M \;(\text{sufficient})$$

second order:

$$\frac{\partial f_0}{\partial t} \simeq C_2(f_0,f_0) + C_0(f_0,f_2) + C_0(f_2,f_0)$$

$$= C_0(f_0, f_2) + C_0(f_0, f_2) + C_2(f_M, f_M)$$

$$= C_\ell(\hat{f}_2) + C_2(f_0, f_0) \qquad \text{--- } \langle 1 \rangle$$

~~Take the~~ Notice $\int d^3v \begin{pmatrix} 1 \\ \frac{1}{2}mv^2 \end{pmatrix} C_\ell(\hat{f}_2) = 0$, proved in prob 3.

1° Take the particle moment of Eq (1).

$$\frac{\partial n}{\partial t} = \int d^3v\, C_\ell(\hat{f}_2) + \int d^3v\, C_2(f_{M0}, f_{M0})$$

$$= \int d^3v\, C_2(f_{M0}, f_{M0})$$

Write as $\frac{\partial n}{\partial t} + \frac{\partial}{\partial X}\Gamma_i = 0$, then Γ_{i0} is defined as

$$\Gamma_i = -\int d^3v\, d^3v'\, D(\underline{v}, \underline{v}')\left(\frac{\partial f_{M0}}{\partial X} f'_{M0} - f_{M0}\frac{\partial f'_{M0}}{\partial X}\right)$$

2°. Take the energy moment of Eq (1).

$$\frac{\partial}{\partial t}\int \frac{1}{2}mv^2 f_M\, d^3v = \int d^3v\, \frac{1}{2}mv^2\, C_2(f_{M0}, f_{M0})$$

i.e.

$$\frac{\partial}{\partial t}\frac{3}{2}nT_i = \frac{\partial}{\partial X}\int d^3v\, d^3v'\, \frac{1}{2}mv^2\, D(\underline{v}, \underline{v}')\left(\frac{\partial f_{M0}}{\partial X} f'_{M0} - f_{M0}\frac{\partial f'_{M0}}{\partial X}\right)$$

$$= \frac{\partial}{\partial X}\frac{5}{2}T_i\int d^3v\, d^3v'\, D(\underline{v}, \underline{v}')\left(\frac{\partial f_{M0}}{\partial X} f'_{M0} - f_{M0}\frac{\partial f'_{M0}}{\partial X}\right)$$

$$+ \frac{\partial}{\partial X}T_i\int d^3v\, d^3v'\left(\frac{mv^2}{2T_i} - \frac{5}{2}\right) D(\underline{v}, \underline{v}')\left(\frac{\partial f_{M0}}{\partial X} f'_{M0} - f_M\frac{\partial f'_{M0}}{\partial X}\right)$$

$$= -\frac{\partial}{\partial X}\frac{5}{2}T_i\Gamma_i - \frac{\partial}{\partial X}q_i$$

with

$$q_i \equiv -T_i \int d^3v\, d^3v' \left(\frac{mv^2}{2T_i} - \frac{5}{2}\right) D(\underline{v},\underline{v}') \left(\frac{\partial f_0}{\partial X} f_0' - f_0 \frac{\partial f_0'}{\partial X}\right)$$

$$\frac{q_i}{T_i} = -A_2$$

From above

$$\Gamma_i = -\int d^3v\, d^3v'\, D(\underline{v},\underline{v}') \left(\frac{\partial f_0}{\partial X} f_0' - f_0 \frac{\partial f_0'}{\partial X}\right)$$

change of variables $\underline{v} \rightleftharpoons v'$, and $D(\underline{v},\underline{v}') = D(\underline{v}',\underline{v})$

$$\text{so } \Gamma_i = -\int d^3v\, d^3v'\, D(\underline{v},\underline{v}') \left(\frac{\partial f_0'}{\partial X} f_0 - f_0' \frac{\partial f_0}{\partial X}\right)$$

$$\text{So } \Gamma_i = -\Gamma_i \implies \boxed{\Gamma_i = 0}$$

$$f_0 = \frac{n}{(2\pi T_i/m)^3} e^{-\frac{mv^2}{2T_i}} \implies \frac{\partial f_0}{\partial X} = \left(A_1 + \left(\frac{m_i v^2}{2T_i} - \frac{5}{2}\right) A_2\right) f_0$$

$$\text{So } \frac{q_i}{T_i} = -\int d^3v\, d^3v' \left(\frac{mv^2}{2T_i} - \frac{5}{2}\right) D(\underline{v},\underline{v}') f_0 f_0' \left(\frac{m_i v^2}{2T_i} - \frac{m_i v'^2}{2T_i}\right) A_2$$

$$\overset{\langle 1 \rangle}{=} -A_2 \int d^3v\, d^3v' \left(\frac{mv^2}{2T_i} - \frac{5}{2}\right) D(\underline{v},\underline{v}') f_0 f_0' \frac{\cancel{T_i}\left(\frac{1}{2}mv^2 - \frac{1}{2}mv'^2\right)}{T_i^2}$$

$$\underset{\langle 2 \rangle}{\overset{\underline{v} \rightleftharpoons \underline{v}'}{=}} -A_2 \int d^3v\, d^3v' \left(\frac{mv'^2}{2T_i} - \frac{5}{2}\right) D(\underline{v},\underline{v}') f_0 f_0' \frac{\frac{1}{2}mv'^2 - \frac{1}{2}mv^2}{T_i^2}$$

$$\frac{\langle 1\rangle + \langle 2\rangle}{2}$$

$$= -A_2 \int d^3v\, d^3v' \frac{\left(\frac{1}{2}mv^2 - \frac{1}{2}mv'^2\right)^2}{2T_i^2} D(\underline{v}, \underline{v}') f_0 f_0'$$

In-terms of the overal transport matrix

$$\begin{bmatrix} \Gamma_i \\ q_i/T_i \end{bmatrix} = \frac{1}{n} \begin{bmatrix} -D & T_{12} \\ T_{21} & -\chi_i \end{bmatrix} \begin{bmatrix} A_1 \\ A_2 \end{bmatrix}$$

Sine A_1 & A_2 are independent to each other $\Gamma_i = 0$

then $D = T_{12} = 0$.

Also we have $T_{21} = 0$ through above calculation or Onsager Symmetry.

$$\chi_i = \frac{n_i}{2T_i^2} \int d^3v\, d^3v' \frac{\left(\frac{1}{2}mv^2 - \frac{1}{2}mv'^2\right)^2}{2} D(\underline{v}, \underline{v}') f_0 f_0'$$

So only χ_i is not zero with

$$D(\underline{v}, \underline{v}') = \underline{e}_y \cdot \underline{\underline{U}} \cdot \underline{e}_y \frac{\Gamma^{ii}}{\Omega^2}$$

Calculation of χ_i

change of variables $(\underline{v}, \underline{v}') \to (\underline{v}_1, \underline{v}_2)$

$\underline{v}_1 = \underline{v} - \underline{v}'$, $\quad \underline{v}_2 = \frac{1}{2}(\underline{v} + \underline{v}')$. therefore

$$\chi_i = \frac{n_i \Gamma^{ii}}{2\Omega^2 T_i^2} \int d^3v \int d^3v' \frac{m_i^2}{|\underline{v}-\underline{v}'|} \left(\frac{1}{2}v^2 - \frac{1}{2}v'^2\right)^2 \left(1 - \frac{(v_y - v_y')^2}{|\underline{v}-\underline{v}'|^2}\right)$$

$$\times \frac{n^2}{(2\pi T_i/m_i)^3} e^{-\frac{m}{2T}(v^2+v'^2)}$$

$$= \frac{n_i \Gamma^{ii}}{2\Omega^2 T_i^2} \frac{n_i^2 m_i^5}{(2\pi T_i)^3} \int d^3v_1 \frac{1}{v_1} \int d^3v_2 \, (\underline{v}_1 \cdot \underline{v}_2)^2 \left(1 - \frac{\underline{e}_y \cdot \underline{v}_1 \underline{v}_1 \cdot \underline{e}_y}{v_1^2}\right)$$

$$\times e^{-\left(\frac{m}{4T}v_1^2 + \frac{m}{T}v_2^2\right)}$$

Note $\quad \frac{1}{(\pi T/m)^{1/2}} \int_{-\infty}^{\infty} dv_x \, v_x^2 e^{-\frac{m}{T}v_x^2} = \frac{T}{2m}$

So $\quad \frac{1}{(\pi T/m)^{3/2}} \int d^3v_2 \, \underline{v}_2 \underline{v}_2 \, e^{-\frac{m}{T}v_2^2} = \frac{1}{2}\frac{T}{m}\underline{\underline{I}}$

So $$\chi_i = \frac{n_i \Gamma^{ii}}{2\Omega^2 T_i^2} \frac{n_i^2 m_i^5}{(2\pi T_i)^3} \frac{1}{2}\frac{T}{m} \underbrace{\int d^3v_1 \, v_1 \left(1 - \frac{v_{1y}^2}{v_1^2}\right) e^{-\frac{m v_1^2}{4T}}} \times \left(\pi\left(\frac{T}{m}\right)\right)^{3/2}$$

$$= \frac{\Gamma^{ii}}{2\Omega^2 T_i^2} \frac{n^2 m_i^5}{(2\pi T_i)^3} \frac{\pi^{3/2}}{2}\left(\frac{T}{m_i}\right)^{5/2} \times \frac{64\pi}{3}\left(\frac{T_i}{m_i}\right)^2$$

Note $\int d^3v_1\, v_1 \left(1 - \frac{v_{1y}^2}{v_1^2}\right) e^{-\frac{mv_1^2}{4T_i}}$

$$= 4\pi \int_0^\infty dv_1\, v_1^3 \left(1 - \frac{1}{3}\right) e^{-\frac{mv_1^2}{3T_i}} = \frac{64\pi}{3}\left(\frac{T_i}{m_i}\right)^2$$

Therefore we get

$$\chi_i = \frac{2n_i}{3\sqrt{\pi}} \frac{\Gamma^{ii}}{\Omega_i^2} \left(\frac{m_i}{T_i}\right)^{\frac{1}{2}} n_i^2$$

Notice $\Gamma^{ii} = \frac{4\pi Z^4 e^4 \ln\Lambda}{m_i^2} = \frac{\nu_{ii}}{n_i}\left(\frac{2T_i}{m_i}\right)^{3/2}$

So $\chi_i = \frac{4\sqrt{2}}{3\sqrt{\pi}} \nu_{ii} \frac{n_i^2 T_i}{m\Omega_i^2}$

From the book $\frac{1}{\tau_i} = \frac{2\sqrt{2}}{3\sqrt{\pi}} \nu_{ii}$, (Eq. 1.5)

So. $\chi_i = 2 \frac{n_i^2 T_i}{m_i \Omega_i^2} \frac{1}{\tau_i}$

This is exactly Braginskii's result Eq 4.48