Lecture 3: Validity of MHD

Ideal MHD Equation

$$\nabla \times \underline{E} = -\frac{\partial B}{\partial t} \qquad \nabla \cdot \underline{B} = 0$$

$$\nabla \times \underline{B} = \mu_0 \underline{J} \qquad n_i = n_e = n$$

$$\frac{\partial \rho}{\partial t} + \nabla \cdot \rho \underline{v} = 0$$

$$\rho \frac{d\underline{V}}{dt} = \underline{J} \times \underline{B} - \nabla p$$

$$\underline{E} + \underline{V} \times \underline{B} = 0$$

$$\frac{d}{dt} \frac{p}{\rho^{r}} = 0$$

Summary of Assumptions

1. Asymptotic: $n_e \to 0,\ c \to 0$
2. Collision dominates: $\left(\frac{m_i}{m_e}\right)^{1/2} \frac{v_{T_i} \tau_{ii}}{a} \ll 1$
   - $\pi \to p$ isotropic
   - equilibration $T_e = T_i$
   - $\kappa \to$ thermal conduction small
3. Small gyro radius: $r_{ii}/a \ll 1$
   - if all term small
   - electron diamagnetism small
   - small terms in energy equation
4. Small resistivity: $\left(\frac{m_e}{m_i}\right)^{1/2} \frac{a}{v_{T_i} \tau_{ii}} \left(\frac{r_{ii}}{a}\right)^2 \ll 1$
   - ηJ in ohms law small
   - Ohmic heating small

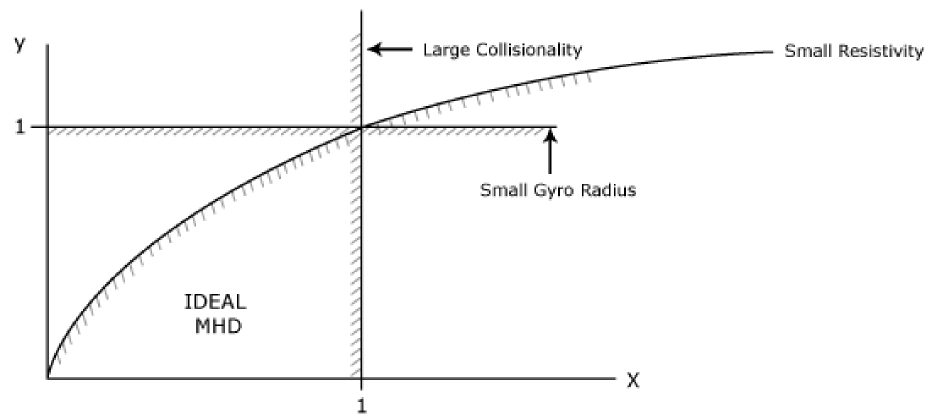
Conditions for validity

1. Define $y = \frac{r_{ii}}{a}$ $\quad x = \left(\frac{m_i}{m_e}\right)^{1/2} \frac{v_{Ti}\tau_{ii}}{a}$

2. Small gyro radius $y < 1$

3. Large collisionality $x < 1$

4. Small resistivity $y^2/x < 1$



Does this regions overlap parameter space of fusion plasmas?

1. Replace y - x diagram as n - T diagram

2. Plasmas of fusion interest

   $10^{18} m^{-3} < n < 10^{20} m^{-3}$

   $\cdot 5\,kev < T < 50\,kev$

3. Rewrite conditions in terms n, T : Note, in this form B and a explicitly appear. Rather than B we hold $\beta = 2\mu_0 nT/B^2$ fixed. β is critical parameter for fusion reactors, set by MHD stability limits.

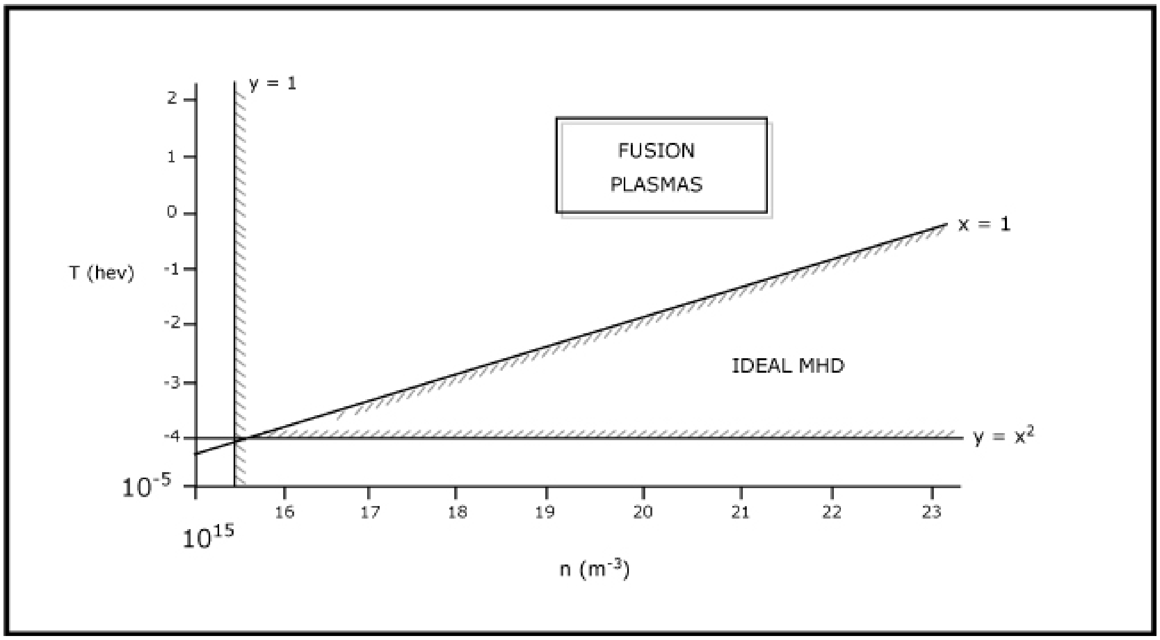
4. Validity conditions $(m \to D, \ln r = 15)$ $n\left(10^{20}\right)$

   a. High collisionality $x = 3 \times 10^3 \left(T^2/an\right) \ll 1$

b. Small gyro radius $y = 2.3 \times 10^{-2} \left(\beta/na^2\right)^{1/2} \ll 1$

c. Small resistivity $y^2/x = 1.8 \times 10^{-7}\, \beta/aT^2 \ll 1$

5. Plot for the case a=1m, $\beta = \cdot 05$



6. Conclusion

Ideal MHD model is not valid for plasmas of fusion interest.

a. Reason- collision dominated assumption breaks down

b. But- large empirical evidence that MHD works very well in describing macroscopic plasma behavior

c. Question- is this lack of subtle physics?

Where specifically does ideal MHD breakdown?

1. Momentum equation

a. $\Pi \ll p$ because of collision dominated assumption

b. $\Pi_\perp \ll p$ from collisionless theory $\Pi_\perp/p \sim r_{ii}/a$ field holds fluid elements together $\perp$ to B.

c. $\Pi_{||} \sim p$ parallel to the field the motion of ions is kinetic
$\tau_{MHD} \sim a/v_{\tau i}$, $\tau_{MIN} \sim a/v_{\tau i}$

d. $\therefore$ $\perp$ momentum equation OK

$\parallel$ momentum equation not accurate

2. Energy Equation

a. $\nabla_{\parallel} K_{\parallel e} J_N T_e \ll \partial p_e / \partial t$ collision dominated assumption

b. $K_{\parallel} \rightarrow \infty$ rather than zero in collisionless plasma

c. More accurate equation of state $\rightarrow \underline{B} \cdot \nabla T = 0$

d. $\therefore$ energy equation not accurate

MHD errors in the momentum and energy equation do not matter why?

1. Momentum $\rho \frac{dV_{\perp}}{dt} = J \times B - \nabla_{\perp} p$ } valid for collisionless

Ohmic law and farday's law $\frac{\partial B}{\partial t} = \nabla \times \underline{V}_{\perp} \times \underline{B}$ } and collisional theory

Note that $\underline{v}_{\parallel}$ does not appears.

2. Errors appear in $\parallel$ momentum equation and energy equation.

3. However, it turns out that for MHD equilibrium and most MHD instabilities, the parallel motion plays a small or negligible role. This is not obvious apriori

4. Assuming this to be true, an incorrect treatment of parallel motion is unimportant, since no parallel motions are exerted: the motions are incompressible.

a. $\underline{B} \cdot \nabla \rho = 0$ no density compression along B

b. $B \cdot \nabla T = 0 \quad \kappa_{\parallel} \rightarrow \infty$

5. The condition $\underline{B} \cdot \nabla \rho = 0$, faradays law and ohms law can be shown to imply $\frac{d\rho}{dt} = 0$. Conservation of mass then implies $\nabla \cdot \underline{v} = 0$

Summary of theories

Collisional	Collisionless	Collisional with $\nabla \cdot \underline{v} = 0$
$\frac{d\rho}{dt} + \rho\nabla \cdot \underline{v} = 0$	$\frac{d\rho}{dt} = 0$	$\frac{d\rho}{dt} = 0$
$\rho \frac{d\underline{v}_\perp}{dt} = \underline{J} \times \underline{B} - \nabla_\perp p$	$\rho \frac{d\underline{v}_\perp}{dt} = \underline{J} \times \underline{B} - \nabla_\perp p$	$\rho \frac{d\underline{v}_\perp}{dt} = \underline{J} \times \underline{B} - \nabla_\perp p$
$\rho \frac{d\underline{v}_\parallel}{dt} = \frac{\underline{B}}{B} \cdot \nabla p \rightarrow$ wrong	$\nabla \cdot \underline{v} = 0$ (equivalent to $\underline{B} \cdot \nabla\rho = 0$)	$\nabla \cdot \underline{v} = 0$
$\underline{E} + \underline{v}_\perp \times \underline{B} = 0$	$\underline{E} + \underline{v}_\perp \times \underline{B} = 0$	$\underline{E} + \underline{v}_\perp \times \underline{B} = 0$
$\frac{d}{dt} p + rp\nabla \cdot \underline{v} = 0 \rightarrow$ wrong	$\underline{B} \cdot \nabla\rho = 0$ (equivalent to $\frac{d\rho}{dt} = 0$)	$\frac{d\rho}{dt} = 0$

Conclusion

a. Once incompressibility is accepted as the dominant motion of unstable MHD modes, then errors in ideal MHD do not enter the calculation.

b. Ideal MHD gives the “same” answer as “collisionless MHD”.

Collisionless derivation from guiding enter theory

$$\underline{J}_\perp = \underline{J}_{mag} + \sum_{\xi i} q_\alpha \int F_\alpha \left[\underline{V}_{\nabla B} + \underline{V}_z + \underline{V}_p + \underline{V}_{\underline{E}\nabla\underline{B}} \right] d\underline{v}$$

$$\underline{J}_{mag} = -\nabla \times \left(\frac{p}{B} \underline{b} \right)$$

MHD ordering

$$\underline{E} + \underline{v}_\perp \times \underline{B} = 0$$

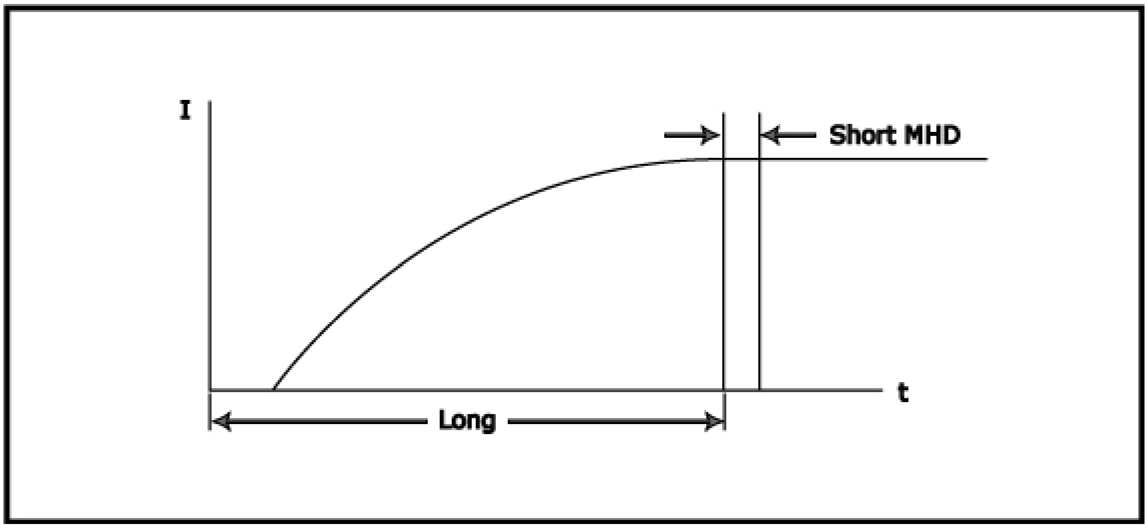
$$\underline{b} \times \left(\frac{d\underline{v}_\perp}{dt} \times \underline{b} \right) = \underline{J} \times \underline{B} - \nabla_\perp p$$

No way to determine equation of state for GC theory

Assume $\frac{dp}{dt} = 0$, $\frac{d\rho}{dt} = 0 \rightarrow$ gives collisionless result.

General Properties of MHD Model

1. Use:



Long time: (transport) $\rightarrow p_\perp \approx p_\parallel \approx$ maxwellion

Short time: continuously test MHD stability as the profile evolves on the slow transport time scale

2. General Conservation Laws

a. Mass $\frac{dM}{dt} = 0 \quad M = \int \rho \, d\underline{r}$

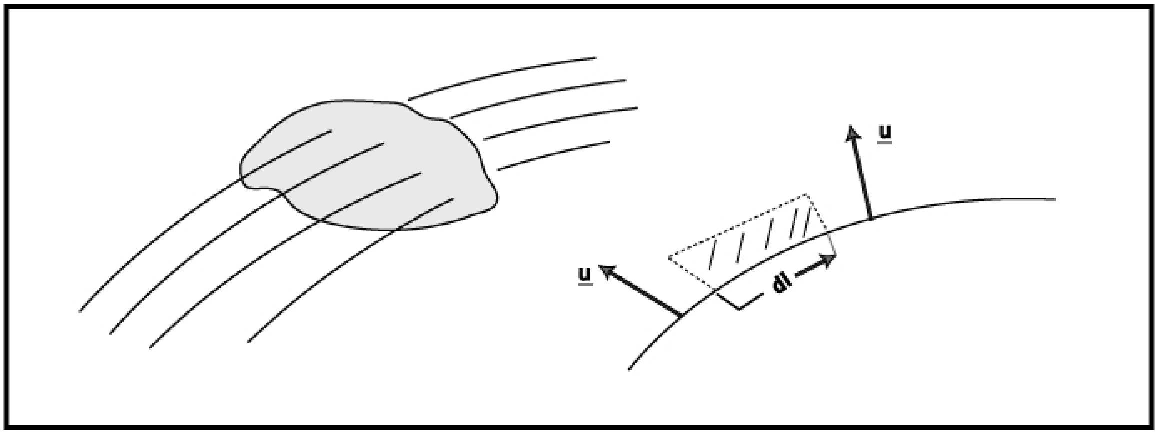
b. Momentum $\frac{d\underline{\underline{P}}}{dt} = 0 \quad \underline{P} = \int \rho \, \underline{v} \, d\underline{r}$

c. Energy $\frac{dW}{dt} = 0 \quad W = \int \left[\frac{1}{2} \rho v^2 + \frac{p}{r-1} + \frac{B^2}{2N_0} \right] d\underline{r}$

Despite approximations, ideal MHD model exactly conserves (3-D nonlinear) mass, momentum and energy.

Conservation of Flux

$\psi = \int \underline{B} \cdot \underline{n}\, ds$



a. $\frac{d\psi}{dt} = \int \frac{\partial B}{\partial t} \cdot \underline{n}\, ds - \int \underline{dl} \cdot \underline{u} \times \underline{B}$

contribution due to motion of surface $\underline{u} =$ arb. surface velocity

$\delta\psi = B dl u \delta t$

$= \underline{B} \cdot (\underline{u} \times \underline{dl}) \delta t$

$\frac{\delta\psi}{\delta t} = -\underline{dl} \cdot \underline{u} \times \underline{B} \rightarrow$ change in ψ due to moving surface.

b. Now $\frac{\partial \underline{B}}{\partial t} = -\nabla \times \underline{E} = \nabla \times \left(\underline{v}_\perp \times \underline{B} - \underline{E}_\parallel\right)$

c. $\frac{d\psi}{dt} = \int \nabla \times \left(\underline{v}_\perp \times \underline{B}\right) \cdot n\, ds - \int \underline{dl} \cdot \underline{u} \times B - \int \nabla \times \underline{E}_\parallel \cdot \underline{n}\, ds$

$= \int dl \cdot \left(\underline{v}_\perp - \underline{u}\right) \times \underline{B} - \int E_\parallel \underline{b} \cdot \underline{dl}$

d. For ideal MHD $E_\parallel = 0$

e. Choose surface motion to coincide with plasma motion: $\underline{u} = \underline{v}$

f. Then

$\frac{d\psi}{dt} = 0$

g. Plasma and field are "frozen" together

h. Important topological constraint: no breaking or tearing of field lines for physical displacements. Topology of $\underline{B}$ lines preserved.

i. Even small resistivity can be important as it allows <u>new</u> motions (tearing modes, resistive interchanges)