22.615, MHD Theory of Fusion Systems
Prof. Freidberg
Lecture 21

Toroidal Tokamak Stability

1. n=0 axisymmetric stability
2. Merceir criterion
3. Ballooning modes several region of stability
4. External kink modes
5. Numerical results (Trogon limit, Sykes limit)

General Comments

1. Toroidal tokamaks quite complicated
2. Equilibria must in general be computed numerically
3. Special high β equilibria calculated in class-unstable because of current jump at the boundary
4. Stability-only Fourier analyze with respect to ϕ

   $$\underline{\xi} = (r, \theta, \phi) = \underline{\xi}(r, \theta)\, e^{-\iota\pi\phi}$$

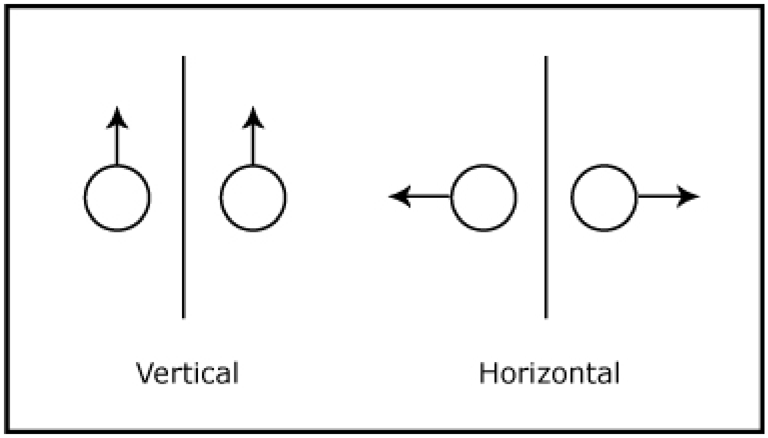
5. Stability equations: 2-D partial differential equations, coefficients function of 2-D equilibria

n=0 axisymmetric modes

1. By symmetry these modes are neutrally stable in the straight case

   $$\delta W(\Lambda = 1)/W_0\Big|_{n=0} = \frac{2}{q_a^2}\left(|m|-1\right) = 0 \text{ for } m = 1$$

2. In a torus we must distinguish vertical from horizontal modes



3. Vertical usually the worst case

4. Simple electrical engineering model

   a. plasma treated as a wire with perfect conductivity

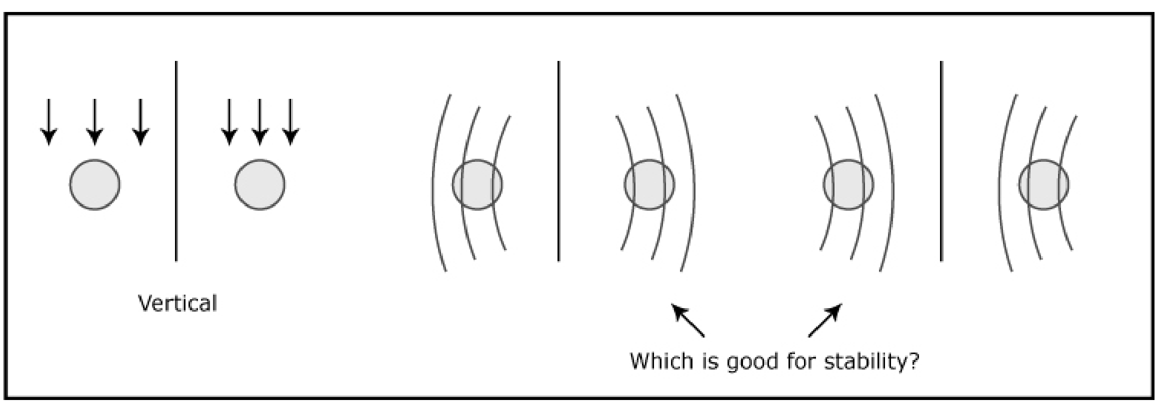
   b. neglect plasma pressure, internal magnetic field, diamagnetism

   c. assume wire is embedded in an external magnetic field

   d. perfect conductivity requires flux within current ring remain constant during the perturbation

   e. goal: calculate shape of the vertical field for stability

5. Pure vertical field is neutral by symmetry



6. Classical mechanics formulation

   a. Force acting on plasma: $\underline{F}(R,Z) = -\nabla\phi$

   b. Equilibrium: $F_R(R_0,Z_0) = F_Z(R_0,Z_0) = 0$

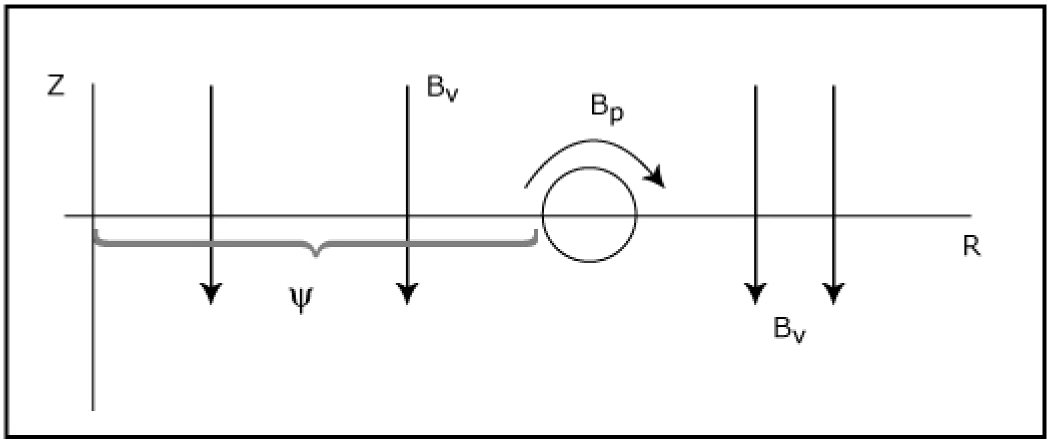
Determines equilibrium position R_0, Z_0 where $\underline{F} = 0$

c. Stability: $\frac{\partial F_Z}{\partial Z}(R_0, Z_0) < 0 \qquad \frac{\partial F_R}{\partial R}(R_0, Z_0) < 0$

vertical stability   horizontal stability

d. Restoring force is opposite to displacement

Formulation



1. Potential Energy: $\phi = \frac{1}{2} L I^2 \qquad L = L(R) = \mu_0 R\left[\ln\left(\frac{8R}{a}\right) - 2\right]$

2. Flux linked by plasma: $\psi = LI - 2\pi \int_0^R B_Z(R', Z) R' dR' = \text{const.}$

   Vertical field

3. Equilibrium: $F_R = F_Z = 0$

$$F_Z = -\frac{\partial \phi}{\partial Z} = -LI\frac{\partial I}{\partial Z} \qquad F_R = -\frac{\partial \phi}{\partial R} = -LI\frac{\partial I}{\partial R} - \frac{I^2}{2}\frac{\partial I}{\partial R}$$

4. Constraint: $\psi = \text{const.} \rightarrow \frac{\partial \psi}{\partial R} = \frac{\partial \psi}{\partial Z} = 0$

a. $\frac{\partial \psi}{\partial Z} = 0 \rightarrow L\frac{\partial I}{\partial Z} - 2\pi \int_0^R \frac{\partial B_Z}{\partial Z} R' dR'$

$$-\frac{1}{R'}\frac{\partial R'}{\partial R'} B_R$$

$$0 = L\frac{\partial I}{\partial Z} + 2\pi R B_R$$

b. $\frac{\partial \psi}{\partial R} = 0 \rightarrow L\frac{\partial I}{\partial R} + I\frac{\partial I}{\partial R} - 2\pi R B_Z = 0$

5. Eliminate $\frac{\partial I}{\partial Z}, \frac{\partial I}{\partial R}$ from force relation

6. $F_Z = 0 \rightarrow 2\pi R I B_R = 0$ $\boxed{B_R\left(R_0, Z_0\right) = 0}$

$$F_R = 0 \rightarrow -\frac{I^2}{2}\frac{\partial L}{\partial R} + I\left(I\frac{\partial L}{\partial R} - 2\pi R B_Z\right) = 0$$

Shafranov result

$$\boxed{B_Z\left(R_0, Z_0\right) = \frac{I}{4\pi R_0}\frac{\partial L}{\partial R_0} = \frac{\mu_0 I}{4\pi R_0}\left[\ln\frac{8R_0}{a} - 1\right]}$$

Vertical Stability

1. $F_Z = -LI\frac{\partial I}{\partial Z} = 2\pi R I B_R$

$$\frac{\partial F_Z}{\partial Z} = 2\pi R\left[\underbrace{B_R}_{0 \text{ from eq.}}\frac{\partial I}{\partial Z} + I\frac{\partial B_R}{\partial Z}\right] = 2\pi R I\frac{\partial B_R}{\partial Z} \quad < 0 \text{ for stability}$$

but $\frac{\partial B_R}{\partial Z} = \frac{\partial B_Z}{\partial R}$ from $\nabla \times B = 0$

define $n\left(R_0, Z_0\right) = -\frac{R_0}{B_Z}\frac{\partial B_Z}{\partial R_0}$ field index

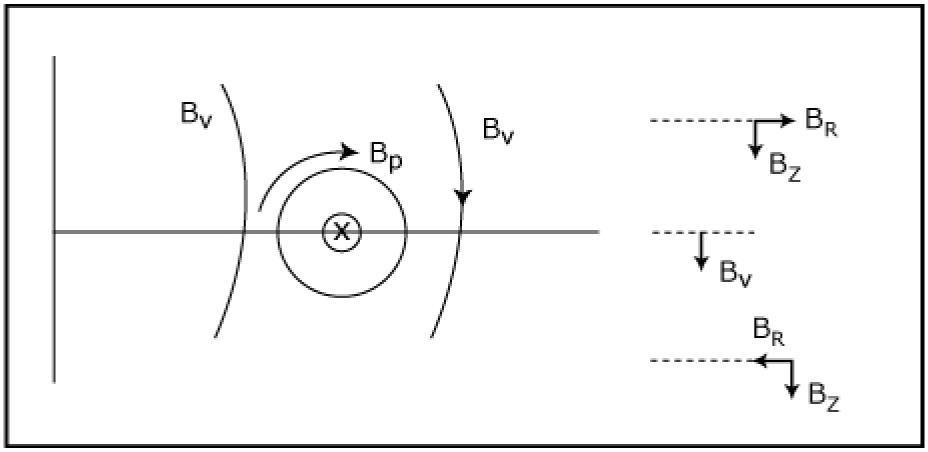
use $B_Z = \frac{I}{4\pi R_0}\frac{\partial L}{\partial R}$ from equilibrium

2. Then vertical stability requires

$$-\frac{I^2}{2R_0}\frac{\partial L}{\partial R_0} n < 0 \text{ or } \boxed{n > 0}$$

defines shape of vertical field

Physical Picture

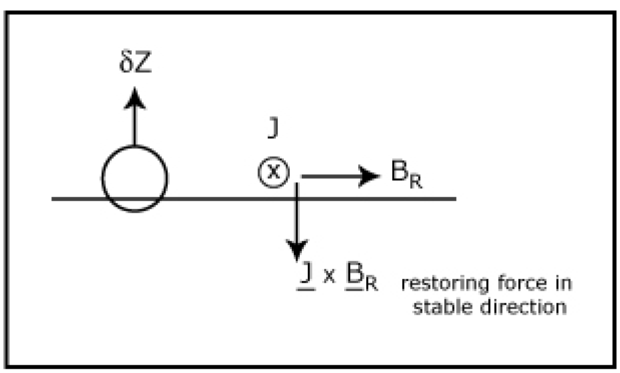


1. with curvature as shown $B_Z < 0$

$$\frac{\partial B_R}{\partial Z} > 0 \quad \left(\frac{\partial B_Z}{\partial R} = \frac{\partial B_R}{\partial Z} \rightarrow \frac{\partial B_Z}{\partial R} > 0 \right)$$

$$\therefore n = -\frac{R}{B_Z} \frac{\partial B_Z}{\partial R} > 0$$

2. give the plasma a vertical displacement



Horizontal Stability

Similar calculation gives

$$\boxed{n \lesssim 3/2}$$

Mercier and Ballooning Modes

1. High n localized internal modes. Competition between line bending and curvature

2. Very localized modes: Suydam 1-D, Mercier 2-D attempt to make line bending as small as possible. Examine remainder of the curvature terms

3. Less localized modes: Ballooning modes important when the curvature oscillates optimally worst eigenfunction. Some line bending, concentration of mode in bad curvature region

4. Plan: Outline derivations of ballooning mode equation and show how Mercier criterion arises.

Starting Point

$$\delta W_F = \frac{1}{2}\int d\underline{r}\left[\frac{\left|Q_\perp\right|^2}{\mu_0} + \frac{B^2}{\mu_0}\left|\nabla\cdot\underline{\xi_\perp} + 2\underline{\xi_\perp}\cdot\underline{\kappa}\right|^2 + \gamma p\left|\nabla\cdot\underline{\xi_\perp}\right|^2\right.$$

$$\left. -2\left(\underline{\xi_\perp}\cdot\nabla p\right)\left(\underline{\xi_\perp^*}\cdot\underline{\kappa}\right) - J_{\parallel}\left(\underline{\xi_\perp^*}\times\underline{b}\right)\cdot\underline{Q_\perp}\right]$$

1. Choose $\xi_{\parallel}$ so $\nabla\cdot\underline{\xi} = 0$ (OK since we assume shear is non-zero)

2. Introduce large n_1 localization assumption by means of an eikonal representation for $\underline{\xi_\perp}$ (similar to WKB)

rapid variation

$$\underline{\xi_\perp} = \underline{n}_\perp e^{\iota S}$$

slow variation (equilibrium scale)

3. Define $\underline{k}_\perp = \nabla S$

$\underline{B}\cdot\nabla S = 0$ S does not vary along B (minimizes line bending)

$$\left|\frac{a\nabla n_\perp}{n_\perp}\right| \sim 1 \qquad \left|a\nabla S\right| \gg 1 \longrightarrow k_\perp \to \infty \text{ limit}$$

4. Evaluate terms: $\underline{Q_\perp} = e^{\iota S}\left[\nabla\times\left(\underline{n}_\perp\times B\right)\right]_\perp$ no ∇S derivatives because $\underline{B}\cdot\nabla S = 0$

5. Then
$$\delta W_F = \frac{1}{2\mu_0}\int d\underline{r}\left[\left|\nabla\times\underline{n}_\perp\times\underline{B}\right|^2 + B^2\left|\iota\underline{k}_\perp\cdot\underline{n}_\perp + \nabla\cdot\underline{n}_\perp + 2\underline{\kappa}\cdot\underline{n}_\perp\right|^2\right.$$
$$\left. -2\mu_0\left(\underline{n}_\perp\cdot\nabla p\right)\left(\underline{n}_\perp^*\cdot\underline{\kappa}\right) - \mu_0 J_{\parallel}\left(\underline{n}_\perp^*\times b\right)\cdot\left[\nabla\times\left(\underline{n}_\perp\times\underline{B}\right)\right]_\perp\right]$$

6. Take the limit $\underline{k}_\perp \to \infty$ and expand

$$\underline{n}_\perp = \underline{n}_{\perp 0} + \underline{n}_{\perp 1} + \ldots, \underline{n}_{\perp 1}\big/\underline{n}_{\perp 0} \sim \frac{1}{k_{\perp a}}$$

7. Zero Order: $\underline{k}_\perp \cdot \underline{n}_\perp = 0 \quad \underline{n}_\perp = Y\,\underline{b}\times\underline{k}_\perp$ slowly varying

   Then $\delta W_0 = 0$

8. Second Order: May Comp. terms:

   only appearance of $\underline{n}_{\perp 1}$

$$\delta W_c = \frac{1}{2\mu_0}\int d\underline{r}\, B^2 \left| \iota \underline{k}_\perp \cdot \underline{n}_{\perp 1} + \nabla\cdot\underline{n}_{\perp 0} + 2\underline{\kappa}\cdot\underline{n}_{\perp 0}\right|^2$$

   Choose $\boxed{\iota \underline{k}_{\perp 0}\underline{n}_{\perp 1} = -\nabla\cdot\underline{n}_{\perp 0} - 2\underline{\kappa}\cdot\underline{n}_{\perp 0}}$ magnetic compressibility does not enter

9. Simple calculation shows that

$$\boxed{\left[\nabla\times\left(\underline{n}_{\perp 0}\times\underline{B}\right)\right]_\perp = \left(\underline{b}\cdot\nabla X\right)\left(\underline{b}\cdot\underline{k}_\perp\right)} \quad X = YB$$

   basic unknown in the problem

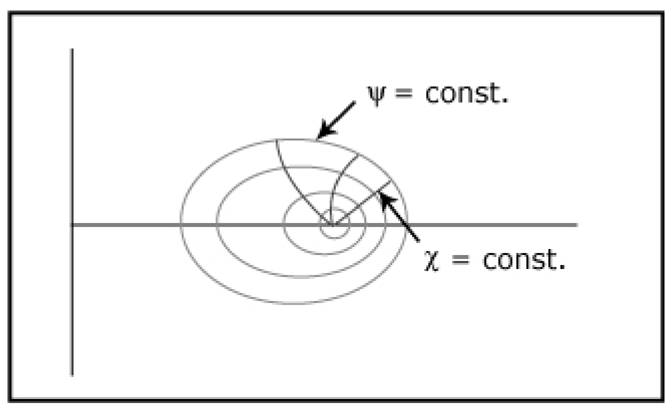
10. Another simple calculation shows that

$$\boxed{J_\parallel\left(\underline{n}_\perp^* \times\underline{b}\right)\cdot\left[\nabla\times\left(\underline{n}_\perp\times B\right)\right]_\perp = 0}$$ kink term does not enter

11. Final δW_F: competition between line bending and field line curvature

$$\boxed{\delta W_2 = \frac{1}{2\mu_0}\int d\underline{r}\left[\underline{k}_\perp^2\left(\underline{b}\cdot\nabla X\right)^2 - \frac{2\mu_0}{B^2}\left(\underline{b}\times\underline{k}_\perp\cdot\nabla p\right)\left(\underline{b}\times\underline{k}_\perp\cdot\kappa\right)|X|^2\right]}$$

Application to tokamaks



1. Flux Coordinates

2. $R, Z \to \psi, \chi$

$$\underline{B} = \frac{F}{R}\underline{e}_\phi + \frac{\nabla\psi\times e_\phi}{R} \qquad \kappa = \underline{b}\cdot\nabla\underline{b}$$

   Choose χ orthogonal $\boxed{\nabla\psi - \nabla\chi = 0}$

3. Then $B_P = \dfrac{|\nabla\psi|}{R}$

$$d\underline{r} = 2\pi R\, dR\, dZ = 2\pi J\, d\psi\, d\chi$$

$$R/J = \underline{e}_\phi\cdot\nabla\chi\times\nabla\psi = R\underline{B}_p\cdot\nabla\chi$$

4. Vector decomposition

$$\underline{n} = \frac{\nabla\psi}{|\nabla\psi|}$$

$$\underline{b} = \frac{B_p}{B}\underline{b}_p + \frac{B_\varphi}{B} e_\varphi \qquad \underline{b}_p = \frac{\underline{B}_p}{B_p}$$

$$\underline{t} = \frac{B_\varphi}{B}\underline{b}_p - \frac{B_p}{B}\underline{e}_\varphi$$

5. Curvature $\underline{\kappa} = \underbrace{\kappa_n}_{\text{normal curvature}} \underline{n} + \underbrace{\kappa_t}_{\text{geodesic curvature}} \underline{t}$

6. $\underline{k}_\perp$ $\qquad \underline{k}_\perp = k_n\underline{n} + k_t\underline{t} = \frac{\partial S}{\partial\psi}\nabla\psi + \frac{\partial S}{\partial\chi}\nabla\chi + \frac{1}{R}\frac{\partial S}{\partial\phi}\underline{e}_\varphi$

$$k_n = \underline{n}\cdot\nabla S = \left(\underline{n}\cdot\nabla\psi\right)\frac{\partial S}{\partial\psi}$$

$$k_t = \underline{t}\cdot\nabla S = \left(\underline{t}\cdot\nabla\chi\right)\frac{\partial S}{\partial\chi} + \left(\underline{t}\cdot\underline{e}_\varphi\right)\frac{1}{R}\frac{\partial S}{\partial\phi}$$

7. $\underline{k}\times\underline{k}_\perp = k_t\underline{n} - k_n\underline{t}$

8. Fourier analysis $\underline{\xi_\perp} \propto \underline{\xi_\perp}(\psi,\chi)\, e^{-\iota n\phi}$

$$\therefore S(\psi,\phi,\chi) = -n\phi + \tilde{S}(\psi,\chi)$$

$$X(\psi,\phi,\chi) = X(\psi,\chi)$$

9. $\underline{b}\cdot\nabla X = \underline{b}\cdot\left[\frac{\partial X}{\partial\psi}\nabla\psi + \frac{\partial X}{\partial\chi}\nabla\chi\right] = \frac{1}{JB}\frac{\partial X}{\partial\chi}$

10. Combine results: $\delta W_2 = \frac{\pi}{\mu_0}\int d\psi\; W(\psi)$

$$\boxed{W(\psi) = \int_0^{2\pi} Jd\chi\left[\left(k_n^2 + k_t^2\right)\left(\frac{1}{JB}\frac{\partial X}{\partial\chi}\right)^2 - \frac{2\mu_0 RB_p}{B^2}\frac{dp}{d\psi}\left(k_t^2\kappa_n - k_t k_n \kappa\right)\right.}$$

a. note that only χ derivatives appears on X

b. stability can be tested one surface at a time !!

Remaining problem: find S

1. $S(\psi,\varphi,\chi)$ satisfies $\underline{B}\cdot\nabla S = 0$ or $\dfrac{B_\varphi}{R}\dfrac{\partial S}{\partial\phi} + \dfrac{1}{J}\dfrac{\partial S}{\partial\chi} = 0$

2. $S = -n\phi + \tilde{S}(\psi,\chi)$ or

   $$S = +n\left[-\phi + \int_{\chi_0}^{\chi}\frac{JB_\varphi}{R}\,d\chi'\right]$$

3. Basic problem with shear and periodicity. Expand about a rational surface $\psi = \psi_0$ for localized modes

   $$S \approx n\left[-\phi + \int_{\chi_0}^{\chi}\left(\frac{JB_\varphi}{R}\right)_{\psi_0} d\chi' + (\psi-\psi_0)\int_{\chi_0}^{\chi}\frac{\partial}{\partial\psi}\left(\frac{JB_\varphi}{R}\right)d\chi'\right]$$

   Rational surface $n\displaystyle\int_{\chi_0}^{\chi_0+2\pi}\left(\frac{JB_\varphi}{R}\right)_{\psi_0} d\chi' = 2\pi m$

4. Problem: $S(\psi,\varphi,\chi) = S(\psi,\phi,\chi+2\pi)$ for periodic solutions. Last term prevents this property since $n(\psi-\psi_0)$ can be finite if $n \gg 1$.

5. Problem resolved by Connor, Hastre and Taylor
   Introduce quasi modes

   $$\underline{\xi}(\psi,\chi) = \sum_p \underline{\xi_\varphi}(\psi,\chi+2\pi p)$$

   $\underline{\xi}$ periodic in $\chi = 2\pi$

   $\underline{\xi_\varphi}$ exists for $-\infty < \chi < \infty$

6. $\underline{\xi_\varphi}$ is not periodic, but if it decays fast enough as $\chi\to\pm\infty$ then $\underline{\xi}$ is periodic

7. Redo entire calculation, almost by inspection.

   $\underline{F}(\psi,\chi)\underline{\xi} = 0 \qquad \underline{F}(\psi,\chi) = \underline{F}(\psi,\chi+2\pi)$

   $$\begin{aligned}\underline{F}(\underline{\xi}) &= \sum_p \underline{F}(\psi,\chi)\xi_\varphi(\psi,\chi+2\pi p)\\ &= \sum_p \underline{F}(\psi,\chi+2\pi p)\xi_\varphi(\psi,\chi+2\pi p)\end{aligned}$$

   $\therefore \underline{\xi_\varphi}$ satisfies $\boxed{\underline{F}(\underline{\xi_\varphi}) = 0}$ same equation as $\underline{\xi}$

8. Whole analysis is now identical except for two points

   $X \to X_\varphi$ quasimode amplitude

$$\int_0^{2\pi} J d\chi \to \int_{-\infty}^{\infty} J d\chi$$

Conclusion:

$$W(\psi) = \int_{-\infty}^{\infty} J d\chi \left[\left(k_n^2 + k_t^2\right)\left(\frac{1}{JB}\frac{\partial X}{\partial \chi}\right)^2 - \frac{2\mu_0 R B_p}{B^2}\frac{dp}{d\psi}\left(k_t^2 \kappa_n - k_t k_n \kappa_t\right) X^2 \right]$$
$$X(-\infty) = X(\infty) = 0$$

$$k_t = \frac{nB}{RB_p}$$

$$k_n = nRB_p \int_{\chi_0}^{\chi} \frac{\partial}{\partial \psi}\left(\frac{JB_\varphi}{R}\right) d\chi'$$

On each surface minimize $W(\psi)$: Find X satisfying $X(-\infty) = X(\varphi) = 0$. Vary χ_0 to find the worst case.