# 22.51 Quantum Theory of Radiation Interactions

### Problem set # 1

Issued on Wednesday Sept. 12 2012. Due on Monday Sept. 24, 2012

## Problem 1: Pauli matrix vector space

**a)** If $A$ and $B$ are matrices of the same dimensions, show that $\langle A, B\rangle = Tr[A^\dagger B]$ has all the properties of an inner product.

**b)** Show that the identity $\mathbb{1}$ and the Pauli matrices $\sigma_x$, $\sigma_y$ and $\sigma_z$ form an orthogonal basis for the vector space of $2\times 2$ matrices with respect to the inner product defined above.

**c)** Write a short program in your favorite language that given a $2\times 2$ matrix will return an expression for the corresponding linear combination of Pauli matrices and identity.

Bonus points: extend this program to expand any $2^n\times 2^n$ square matrix onto the Pauli basis (a basis composed by all possible combinations of Pauli matrices for the $n$ TLS, e.g. $\{\mathbb{1}, \sigma_x^1, \dots, \sigma_x^2, \dots, \sigma_x^1\otimes\sigma_x^2, \sigma_x^1\otimes\sigma_y^2, \dots \sigma_z^1\sigma_z^2\}$ for two TLS).

## Problem 2: Schwartz's inequality and uncertainty relationship

**a)** The simplest way to derive the Schwartz's inequality goes as follows. First, observe that

$$(\langle\alpha| + \lambda^*\langle\beta|)\cdot(|\alpha\rangle + \lambda|\beta\rangle) \geq 0$$

for any complex number $\lambda$; then choose $\lambda$ in such a way that the preceding inequality reduces to the Schwartz's inequality.

**b)** Relate $|\alpha\rangle$ and $|\beta\rangle$ conveniently to $\Delta A$ and $\Delta B$ ($\Delta A = A - \langle A\rangle$; similarly for $B$), and prove the uncertainty relation for two observables $A$, $B$

$$\langle\psi|(\Delta A)^2|\psi\rangle\langle\psi|(\Delta B)^2|\psi\rangle \geq \frac{1}{4}|\langle\psi|[A, B]|\psi\rangle|^2$$

for an arbitrary state $|\psi\rangle$.

**c)** Show that the equality sign in the uncertainty relation holds if the state in question satisfies $\Delta A|\psi\rangle = a\Delta B|\psi\rangle$ with $a$ purely imaginary.

## Problem 3: Bloch Sphere and Diagonalization

**a)** Show that a general $2\times 2$ hermitian matrix

$$A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$$

can be diagonalized by the operator $U = e^{i\phi\sigma_z/2}e^{i\theta\sigma_y/2}$, by giving first an intuitive (geometric) reason and then the explicit conditions on $\phi$, $\theta$.

## Problem 4: Exponential of an operator

Consider the operator $A = |\psi_1\rangle \langle\psi_2|$, where $|\psi_i\rangle$ are orthonormal kets in $\mathbb{C}^2$.

**a)** What is $e^{xA}$ (where $x$ is a complex number)?

**b)** Is $e^{xA}e^{xA^\dagger} = e^{x(A+A^\dagger)}$ ? (write also an explicit expression for the exponentials).

## Problem 5: Repeated Measurements

Two operators $A$ and $B$ describe observables on a system initially in the state $|\psi\rangle$. We perform successive measurements on the system (first $A$ then $B$ and finally $A$ again). The operator $A$ has normalized eigenkets $|a_k\rangle$ $(k = \{1, 2, 3\})$. The eigenvectors of $B$ are related to the eigenvectors of $A$ by the relationship: $|b_1\rangle = (3\,|a_1\rangle + 5\,|a_2\rangle)/\sqrt{34}$, $|b_2\rangle = (3\,|a_2\rangle - 5\,|a_1\rangle)/\sqrt{34}$ and $|b_3\rangle = |a_3\rangle$.

**a)** What is the probability of obtaining $a_3$ as the result of the third measurement?

**b)** Assume $A$ is measured and the result $a_1$ is obtained. What is the probability of obtaining $a_2$ in the third measurement?

## Problem 6: Quark Oscillations

We saw in class that neutrinos can oscillates between different flavors because of the mixing of their mass eigenstate. A similar phenomenon occurs for quark. Consider for example the strange and down quark as two states of the quark.

The most general state vector can then be written as

$$|\alpha\rangle = \alpha\,|d\rangle + \beta\,|s\rangle\,.$$

The quark can change its state from down to strange.

**a)** What is the Hamiltonian $H$ describing the interaction that makes the quark go from the down to the strange flavor and vice-versa?

**b)** Find the normalized energy eigenkets of the Hamiltonian. What are the corresponding energy eigenvalues?

**c)** Suppose a quark is created in the state $|\alpha\rangle$ as given above at $t = 0$. Find the state vector at $t > 0$ by applying the appropriate time-evolution operator to $|\alpha\rangle$.

**d)** Suppose at $t = 0$ a down quark is produced in a reaction. What is the probability for observing strange quark as a function of time?

**e)** Suppose I suggested you that the (wrong) answer to question (a) is $H = \Delta|s\rangle\langle d|$. By explicitly solving the time-evolution problem with this Hamiltonian, show that probability conservation is violated.

**f)** Now assume that quark of different flavors have different energies $\omega_d > \omega_s$. Write the Hamiltonian describing this energy difference in the $|d\rangle\,, |s\rangle$ basis.

**g)** Calculate again the evolution of a down quark (as in question (c)) when to the Hamiltonian you found in question (a) you add the Hamiltonian of question (f).

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