22.101 Fall 2003 Professor Sow-Hsin Chen Due: Nov. 3, 2003.

Problem Set No.4

1. You are asked to derive in detail the low energy theorem for scattering of neutrons by a potential well.
The potential well has a range b and a depth of $-V_0$. Recall from the bound-state problem for the deuteron, we have the following relations:

$$\kappa b \cot \kappa b = -\alpha b, \qquad \text{where } \kappa^2 = \frac{M}{\hbar^2}(V_0 - B) = K_0^2 - \alpha^2 \qquad (1)$$

and $\alpha = \sqrt{MB/\hbar^2} = 0.232\ (\text{fm})^{-1}, \quad b \le 3\ \text{fm}$.

In the scattering problem, the total energy is the center of mass neutron kinetic energy, $E = \frac{\hbar^2 k^2}{M}$. The consistency relation from matching the radial wave function at the edge of the well is

$$k \cot(kb + \delta) = K \cot Kb, \qquad (2)$$

where

$$K^2 = K_0^2 + k^2. \qquad (3)$$

(a) By expanding the cotangent function on the left-hand side of Eq.(2), show that

$$k \cot \delta = \frac{K \cot Kb + k \tan kb}{1 - \frac{K}{k} \tan kb \cot Kb}. \qquad (4)$$

(b) Taking k = 0 limit of Eq.4, and derive a relation

$$K_0 \cot K_0 b = \frac{1}{b - a}. \qquad (5)$$

(c) In the small k limit, expand Eq.1 as

$$K_0 \approx \kappa + \frac{\alpha^2}{2\kappa} \qquad (6)$$

and Tayler expanding the function $K_0 \cot K_0 b$ around κ to show that

$$K_0 \cot K_0 b = \kappa \cot \kappa b + \frac{\alpha^2}{2\kappa}\left(\cot \kappa b - \kappa b \csc^2 \kappa b\right) = -\alpha - \frac{1}{2}\alpha^2 b. \qquad (7)$$

(d) Combining Eq.7 and Eq.(5) and show

$$a = b + \frac{1}{\alpha}(1 - \frac{1}{2}\alpha b) \qquad (8)$$

This last relation shows that a small binding energy of the bound state will result in a large scattering length for the low energy scattering.

(e) Again using the approximation valid in the low energy limit $K \approx K_0 + \frac{k^2}{2K_0}$ to show that

$$K \cot Kb = K_0 \cot K_0 b + \frac{k^2}{2K_0}\left(\cot K_0 b - K_0 b \csc^2 K_0 b\right) = \frac{1}{b-a} - \frac{1}{2} bk^2 - \frac{ak^2}{2K_0(b-a)^2}. \qquad (9)$$

(f) Combining Eq.4 and eq.9, to show finally the very useful low energy theorem

$$k \cot\delta = \frac{1}{a} - \frac{1}{2}(b - \frac{b^3}{3a^2} - \frac{1}{K_0^2 a})k^2. \qquad (10)$$

Eq.10 means that for neutron energy E below 10 MeV, we have a cross section of the form

$$\sigma(k) = 4\pi a^2\left\{1 - k^2\left[a(a-b) + \frac{b^3}{3a} - \frac{1}{K_0^2}\right]\right\}, \qquad (11)$$

which shows clearly that for thermal neutrons the total cross-section is simply a constant value $4\pi a^2$, a very important result.

2. Show that in the limit $k \to 0$, the total $n-p$ scattering cross section can be written as

$$\sigma(k \to 0) = 4\pi b^2\left(1 - \frac{\tan K_0 b}{K_0 b}\right)^2. \qquad (12)$$

where b is the range of force in the square well potential, and $K_0 = \left(MV_0/\hbar^2\right)^{1/2}$.

3 Solve the S-wave scattering of a slow particle $(kb << 1)$ by a repulsive potential barrier.

$$V(r) = \begin{cases} +V_0 & r < b \\ 0 & r > b \end{cases}. \qquad (13)$$

Derive the scattering length a in terms of the potential parameters in the limit of zero energy. Derive the total cross section and show that as $V_0 \to \infty$ the total cross section approaches a limit $4\pi b^2$, which is four times the geometrical cross section of a hard sphere of radius b.

4. In order to perform a proton-proton scattering experiment at at $E_L = 10$ MeV, one fills the scattering chamber with 10 mmHg of H_2 gas at $0^\circ C$. The proton beam from a cyclotron has a diameter of 2 mm and has been calibrated to have a current of 5 μA. One detects the scattered proton beam at $\Theta_L = 45^\circ$ by a detector which subtends a solid angle of 10^{-3} sterad. Assume the scattering volume of $(0.2)^3\,cm^3$ and the differential cross section at this angle of 0.17 barns/sterad. Calculate the counting rate at the detector.

5. The fission cross section for thermal neutrons of natural uranium (isotopic composition $0.72\%\,U^{235}$, $99.27\%\,U^{238}$) is $\sigma_f = 4.22\,b$. Calculate the fission cross section of pure U^{235} for thermal neutrons. If a thermal neutron flux $\phi = 10^{10}$ neutrons/cm^2 sec impinges on a thin target of natural uranium, how many fission events occur per second when the target is 0.01 cm thick? (Density of uranium $\rho = 18.68\ gr/cm^3$).