Equations

- Slice Selection: $ideal\ \ O_{2D}(x,y) = \int_{\Omega} O_{3D}(x,y,z)\delta(z - z_s)dz$
- Better Approximation: $O_{2D}(x,y) = \int_{\Omega} O_{3D}(x,y,z) TopHat\left(\frac{z - z_s}{\frac{1}{2}\Delta z}\right)dz$
- Radon Transform: $P(\theta,z) = \int \int O(x,y)\delta(x\cos(\theta) + y\sin(\theta) + z)$
- Fourier Transform: $\tilde{g}(k) = \int G(x)e^{-ikx}dx$
- Central Slice Theorem: $k_z = k_x\cos(\theta) + k_y\sin(\theta)$

  $z = x\cos(\theta) + y\sin(\theta)$
- Back Projection: $O_r(x,y) = \frac{1}{\pi}\int_0^{\pi} P(\theta,z)d\theta$
- Map: $\tilde{\tilde{O}}_r(k_x,k_y) = \tilde{P}(\theta,k_z)$

  $where\ \ k_z = k_x\cos(\theta) + k_y\sin(\theta)$

More Organized Proof of The Central Slice Theorem

The PSF associated with the simple Bach projection is:

$$PSF\big|_{BF} = \frac{1}{r}$$

$$\therefore \quad O_r(x,y) = O(x,y) \otimes \frac{1}{\sqrt{x^2 + y^2}}$$

where $O_r(x,y) = B\{P(\alpha,z)\}$

and $B = \frac{1}{\pi}\int_0^R P(\alpha_1 x\cos(\alpha) + y\sin(\alpha))d\alpha$

$$\therefore \quad \underbrace{O_r(x,y)}_{\Updownarrow} = \underbrace{O(x,y)}_{\Updownarrow} \otimes \underbrace{\frac{1}{\sqrt{x^2 + y^2}}}_{\Updownarrow}$$

$$\tilde{\tilde{O}}_r(k_x,k_y) = \tilde{\tilde{O}}(k_x,k_y) \bullet \frac{1}{|k|}$$

so

$$\tilde{\tilde{O}}(k_x,k_y) = |k| \; \tilde{\tilde{O}}_r(k_x,k_y)$$

More Organized Proof of The Central Slice Theorem

1. $P(\alpha,z) = \iint O(x,y)\delta(x\cos(\alpha) + y\sin(\alpha) - z)dxdy$

2. Equate the z− axis with a tilted reference frame

   $x^{'} \parallel z, \; y^{'} \perp z$

   $\therefore \quad x = x^{'}\cos(\alpha) - y^{'}\sin(\alpha)$

   $y = x^{'}\sin(\alpha) + y^{'}\cos(\alpha)$

   and

   $x^{'} = x\cos(\alpha) + y\sin(\alpha)$

3. Substitute #2 into #1 and change integral to $dx^{'}dy^{'}$ (still over all space)

   $P(\alpha,z) = \iint O(x^{'}\cos(\alpha) - y^{'}\sin(\alpha), x^{'}\sin(\alpha) + y^{'}\cos(\alpha))\delta(x^{'} - z)dx^{'}dy^{'}$

4. Integrate along $x^{'}$ and note that z is only a point along the $x^{'}$ axis.

   $P(\alpha,x^{'}) = \iint O(x^{'}\cos(\alpha) - y^{'}\sin(\alpha), x^{'}\sin(\alpha) + y^{'}\cos(\alpha))dy^{'}$

5. Fourier Transform along $x^{'}$

   $\tilde{p}(\alpha,k_{x^{'}}) = \iint O(x^{'}\cos(\alpha) - y^{'}\sin(\alpha), x^{'}\sin(\alpha) + y^{'}\cos(\alpha))e^{-ix^{'}k_{x^{'}}}dx^{'}dy^{'}$

More Organized Proof of The Central Slice Theorem

6. Transform back to the (x,y) coordinate system

$$\tilde{p}(\alpha,k_{x'}) = \iint O(x,y) e^{-i(x\cos(\alpha)+y\sin(\alpha))k_{x'}} dxdy$$

7. Define the tilted k− space coordinate system.

$$k_x = k_{x'}\cos(\alpha) - k_{y'}\sin(\alpha)$$

$$k_y = k_{x'}\sin(\alpha) - k_{y'}\cos(\alpha)$$

8. Rewrite #6 as

$$\tilde{p}(\alpha,k_{x'}) = \iint O(x,y) e^{-i\left(k_{x'}\cos(\alpha)-k_{y'}\sin(\alpha)\right)x} e^{-i\left(k_{x'}\sin(\alpha)+k_{y'}\cos(\alpha)\right)y} dxdy \Big|_{k_{y'}=0}$$

$$\tilde{p}(\alpha,k_z) = \iint O(x,y) e^{-ik_x x} e^{-ik_y y} dxdy \Big|_{k_{y'}=0}$$

$$= F_{2D}\{O(x,y)\}\Big|_{k_{y'}=0}$$

The Central Slice Theorem

Consider a 2-dimensional example of an emission imaging system. O(x,y) is the object function, describing the source distribution. The projection data, is the line integral along the projection direction.

$$P(0^\circ, y) = \int O(x,y)\,dx$$

The Central Slice Theorem can be seen as a consequence of the separability of a 2-D Fourier Transform.

$$\tilde{o}(k_x, k_y) = \int O(x,y) e^{-ik_x x} e^{-ik_y y}\,dxdy$$

The 1-D Fourier Transform of the projection is,

$$\begin{aligned}
\tilde{p}(k_y) &= \int P(0^\circ, y) e^{-ik_y y}\,dy \\
&= \int O(x,y) e^{-ik_y y}\,dxdy \\
&= \int O(x,y) e^{-ik_y y} e^{-i0x}\,dxdy \\
&= \tilde{o}(0, k_y)
\end{aligned}$$

The Central Slice Theorem

The one-dimensional Fourier transformation of a projection obtained at an angle J, is the same as the radical slice taken through the two-dimensional Fourier domain of the object at the same angle.