22.02 Intro to Applied Nuclear Physics

Mid-Term Exam

Solution

Problem 1: Short Questions *24 points*

*These short questions require only **short answers** (but even for yes/no questions give a brief explanation)*

a) *In the SEMF, the volume term gives the most important contribution to the binding energy B, setting $B \propto A$ (with A the mass number). What does this tell us about the nuclear force keeping nucleons together?*
[Hint: your answer – and the word! – would be much different if the volume term were $B \propto A(A-1)$]

Solution:

This behavior of the binding energy tells us that the nuclear force has a short range so that each nucleon only interacts with a fixed, small number of neighbors and not with all the other $A-1$ nucleons.

b) *True or False?:*
i) The Q-value of a fusion reaction is always $Q > 0$ (and very large!).
ii) Still it is very difficult to obtain fusion because of the Coulomb repulsion V_C between the fusing nuclei.
ii) Thus we always need to give the fusing nuclei enough energy that their energy is $E > V_C$, by going to very high temperatures.

Solution:

The first statement is true only for two light nuclei fusing together (in general two nuclides at the left of the maximum in the B/A curve). The second statement is true, while the last one is false: we need to give enough energy for the two fusing nuclide to *tunnel* through the coulomb barrier.

c) *What is an observable in quantum mechanics? What mathematical object represents it? (Give also an example).*

Solution:

An observable is a physical property that we can measure; it is represented by an hermitian operator. Examples are momentum, position, magnetic moment, energy...

d) *What is a complete set of commuting observables and why do we need it?*
Are $\{L_{xy},\ L_z\}$ a complete set of commuting observables? (Here $L_{xy} = L_x^2 + L_y^2$)

Solution:

A complete set of commuting observables is a set of observables (operators) that commute and that can completely lift any degeneracy in their common eigenfunctions, so that the state eigenfunction is completely determined.

The fact that L_x, L_y do not commute with L_z does not imply that L_{xy} does not commute with L_z; in fact the opposite can be very easily proven by just knowing the commutation of L_z with L^2. Note that $L_{xy} = L^2 - L_z$. Since both L^2 and L_z commute with L_z, L_{xy} commute with L_z: thus $\{L_{xy},\ L_z\}$ can be part of a complete set of commuting observables. Indeed they do form a complete set commuting observables: as for L^2, eigenfunctions of L_{xy} are doubly degenerate, since for each eigenvalue of L_{xy}, given by $\hbar^2[l(l+1) - m_z^2]$ there are two possible eigenfunctions $|l, m_z\rangle$ and $|l, -m_z\rangle$. Measuring both L_{xy} and L_z determines completely the eigenfunction.

e) *If we measure the potential energy $\hat{V} = V(\hat{x})$ of a quantum particle and immediately after we measure its position $\hat{x}$, is the result of the second measurement random?*

Solution:

No: if the potential energy measurement gave an outcome $V(x^\star)$, then the position is $x^\star$ with probability 1. This is because the potential energy measurement projects the wavefunction of the system into a potential energy eigenfunction (wavefunction collapse) and this potential energy eigenfunction is also an eigenfunction of the position, since the two operators commute. (The two operators commute because $V(\hat{x})$ is a function of $\hat{x}$).

f) *Which one of the following statements (if any) is correct, based on the properties of the angular momentum and its eigenfunctions?*
1) A particle is in the angular momentum eigenstate: $\psi_{l,m_z}(\vartheta,\varphi) = |l\text{=-}1, m_z\text{=}0\rangle$.
2) A particle is in the angular momentum eigenstate: $\psi_{l,m_z}(\vartheta,\varphi) = |l\text{=}0, m_z\text{=-}1\rangle$.
3) A particle is in the angular momentum eigenstate: $\psi_{l,m_z}(\vartheta,\varphi) = |l\text{=}1, m_z\text{=}0\rangle$.
4) A particle is in the angular momentum eigenstate: $\psi_{l,m_y}(\vartheta,\varphi) = |l\text{=}1, m_y\text{=-}1\rangle$.

Solution:

Wrong (l is always a non-negative integer). Wrong ($-l < m_z < l$ in integer steps). Correct, Correct: both these last two states respect all the requirements for l and $|m| \leq l$, both integers.

g) *Consider a finite quantum well of depth V_w and width $2a$ (between $-a$ and $+a$, while $V = 0$ outside the well) and a particle in an even energy eigenfunction with energy E. Is the particle bound?*

Solution:

Since the solution is an even solution, we can get a bound state for any value of a and V_w. However, we have a bound state only if $E < 0$. If $E > 0$ the particle is just scattering above the well. Note also that if $E < V_w$ we don't have a solution at all.

h) *A system is subject to a potential $V(x)$ and is found in two possible states described by two different wavefunctions. The first wavefunction is an eigenfunction of the momentum while the second wavefunction is an eigenfunction of the Hamiltonian. Which wavefunction describes a stationary state? Why?*

Solution:

The energy eigenfunction because it is a solution of the time-independent Schrödinger equation. Do not confuse the momentum eigenfunction with the momentum itself: the momentum is an operator. In the description of QM we have until now, operators do not evolve in time (although their expectation value and measurement outcomes do, since they also depend on the system state which does depend on time). Thus, although the momentum operator is not evolving in time, its eigenfunctions do evolve and are not stationary states.

Problem 2: Temperature *13 points*

a) *In classical thermodynamics, the temperature of an ideal gas (in 1D) is given by $T = \frac{m}{k_B}v^2$ (where k_B is the Boltzmann constant, m the mass and v the velocity). Define the corresponding quantum mechanical observable in terms of observables seen in class.*

Solution:

$T = \frac{1}{mk_b}(mv)^2 = \frac{1}{mk_b}p^2$. Transforming p into the QM operator, we have

$$\hat{T} = -\frac{\hbar^2}{mk_b}\frac{d^2}{dx^2}$$

b) *Consider a "gas of particles" inside an infinite well ($V(x) = 0$ for $0 < x < L$ and infinite otherwise). Each particle is in the state*

$$\psi(x) = \sqrt{\frac{1}{2L}}\sin\left(\frac{3\pi x}{L}\right) + \sqrt{\frac{1}{L}}\sin\left(\frac{10\pi x}{L}\right) + \sqrt{\frac{1}{2L}}\sin\left(\frac{11\pi x}{L}\right)$$

What is the expectation value (average) of the temperature?
[Note: you should be able to solve this problem without doing any integral!]

Solution:

We recognize the eigenfunctions of the Hamiltonian $\varphi_n(x) = \sqrt{\frac{2}{L}}\sin\left(\frac{n\pi x}{L}\right)$: note the factor $\sqrt{\frac{2}{L}}$ which makes sure the eigenfunction is correctly normalized. If you did not remember this, you should at least considered that the sum of proabilities for the 3 states should have given you 1.
Thus the state can be written:

$$\psi(x) = \frac{1}{2}\varphi_3 + \frac{1}{\sqrt{2}}\varphi_{10} + \frac{1}{2}\varphi_{11}$$

These are also eigenfunctions of the temperature (inside the well) since the temperature and the Hamiltonian (equal to the kinetic energy in the well) commute. The corresponding temperature eigenvalues are

$$T_n = \frac{2}{k_b}E_n = \frac{\hbar^2 k^2}{mk_b} = \frac{\hbar^2\pi^2}{L^2 mk_b}n^2.$$

The average temperature is then:

$$\langle T\rangle = \sum_n T_n P_n = \sum_n T_n |c_n|^2 = \frac{\hbar^2\pi^2}{L^2 mk_b}\left(\frac{1}{4}3^2 + \frac{1}{2}10^2 + \frac{1}{4}11^2\right) = \frac{\hbar^2\pi^2}{2L^2 mk_b}165 = \frac{\hbar^2\pi^2}{L^2 mk_b}82.5$$

Some of you have started from $\hat{T} = \frac{1}{mk_b}\hat{p}^2$ and calculated $\left\langle \hat{T}\right\rangle = \frac{1}{mk_b}\langle\hat{p}\rangle^2$: this is wrong since in general $\langle p^2\rangle \neq \langle p\rangle^2$ (also for any other observable and any random variable).

Problem 3: Scattering *12 points*

A source of neutrons produces a flux of neutrons of intensity Φ_{inc} and energy E. To protect the worker, in front of the source a wall of thickness L has been built, which we can model with a potential barrier $V > E$. To monitor her health, the worker wears a detector that measure the neutron flux.

a) *Sketch a simple drawing showing how you model this problem (assume we leave in a 1D world!). Show in the sketch the characteristics of the wavefunction describing the neutron.*

Solution:

Note: this problem does not describe how neutron shielding does work in practice. We will see that neutrons undergo many scattering events in the material, being slow down in the wall.

For the problem at hand, the situation is describe by a traveling wave before and after the wall (with same wavelength λ but different amplitude) and a decaying exponential in the wall.

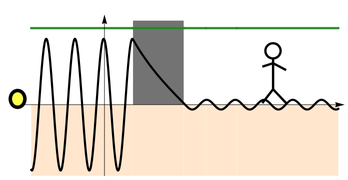


Figure 1: Sketch of neutron shielding

b) *What is the flux of neutrons measured by the detector? Assume a simple model for the tunneling, in the limit where the tunneling probability is small: this means that you do not need to solve completely the problem to find the –approximate– answer.*

Solution:

The detector measures the transmitted flux. In the limit of small tunneling, we can write $\Phi_t \approx 4e^{-2\kappa L}\Phi_{inc}$, with $\kappa = \sqrt{\frac{2m(V-H)}{\hbar^2}}$.

Problem 4: Spontaneous Fission *30 points*

Useful quantities: $\hbar c = 197$*MeV fm;* $\frac{e^2}{\hbar c} = \frac{1}{137}$*;* $c = 3\times 10^8$*m/s;* $R_0 = 1.25$*fm.*

A note about this problem: Because of the small and large quantities involved, small rounding errors give very different results. This is ok in the Midterm, but if you want to calculate something like this in real life, make sure you pay attention to numerical errors.

Also, although the final answer does give you an idea about the fact that Cm can indeed undergo spontaneous fission, estimating the fission rate in this way is a bad approximation, since we expect Cm to fission not in just these two fragments but to lead to a distribution of possible pairs of fragment, all accompanied by the release of some neutrons. This would be a 3 (or larger) body problem, which complicates estimates.

a) *Consider the isotope Curium-250 ($^{250}_{96}$Cm), with mass 232.938 GeV. Given its A and Z numbers, do you expect this isotope to be stable?*

Solution:

The ratio of the mass and proton number is $Z/A \approx 0.384$. We know that for heavy stable nuclei this ratio is instead ≈ 0.41 ($Z \approx A/2$ only for light nuclides). Thus we expect this isotope to be unstable and to decay by a process that will make it shed some neutrons or acquire smaller A.

Curium-250 is the lightest nuclide to undergo spontaneous fission as the main decay mode. We want to analyze this decay mode following the same theory we saw for alpha decay and in particular *estimate* the half-life of ^{250}Cm. The following questions will guide you through the estimation.

b) *Assume that the spontaneous fission leads to the decay:*

$$^{250}_{96}Cm \rightarrow \, ^{130}_{52}Te_{78} + ^{120}_{44}Ru_{76}$$

What is the Q-value for the reaction? How does this compare to the usual Q-value for typical alpha decay?
[The mass of $^{250}_{96}$Cm is 232.938GeV, the mass of $^{130}_{52}Te_{78}$ is 121.002 GeV and the mass of $^{120}_{44}Ru_{76}$ is 111.724GeV]

Solution:

The Q-value can be calculated from the mass differences:

$$Q = m_{Cm} - m_{Nd} - m_{Kr} = 0.212GeV = 212MeV$$

Typical alpha decay release a few MeV of energy, so the Q value here is much larger.

c) *What is the Coulomb potential $V_C(R) = \frac{Q_1 Q_2}{R}$ at the distance $R = R_{Te} + R_{Ru}$, where R_X is the nuclear radius? What is the distance R_c at which the Coulomb potential is equal to the Q-value?*

Solution:

The nuclear radius is $R_A = 1.25A^{1/3}$fm, thus we have:

$$R_{Ru} = 1.25fm \times 120^{1/3} = 6.17fm \qquad\qquad R_{Te} = 1.25fm \times 130^{1/3} = 6.33fm$$

yielding $R = 12.50$fm.
The Coulomb potential is given by

$$V_C(R) = \frac{e^2 Z_{Nd} Z_{Kr}}{R} = \frac{e^2}{\hbar c}\hbar c \frac{Z_{Nd} Z_{Kr}}{R} = \frac{1}{137} 197\text{MeV fm} \frac{44 \times 52}{12.50\text{fm}} = 263.20\text{MeV}$$

Note: because the fine constant $\frac{e^2}{\hbar c} = \frac{1}{137}$ was given in cgs units, the Coulomb potential could be calculated from it in cgs units.
To find the distance R_c we equate the Coulomb potential to the Q-value:

$$\frac{e^2 Z_1 Z_2}{R_c} = Q \quad \rightarrow \quad R_C = \frac{e^2 Z_1 Z_2}{Q} = R\frac{V_C(R)}{Q} = 12.50\text{fm} \times \frac{263.20}{212} = 15.52\text{fm}$$

d) *We want to estimate the probability of tunneling for an effective particle in the center of mass frame of the Tellurium and Ruthenium nuclei (with reduced mass μ).*
To estimate the tunneling probability we replace the Coulomb barrier with a rectangular barrier of height $V_H = V_C(R)$ and length $L = (R_c - R)$. What is the tunneling probability?

Solution:

The reduced mass is

$$\mu = \frac{m_1 m_2}{m_1 + m_2} = \frac{121.002 \times 111.724}{111.724 + 121.002} = 58.089GeV$$

Since we want to *estimate* the tunneling probability, we take the approximate expression $P_T = 4e^{-2\kappa L}$. We first need to calculate κ:

$$\kappa = \frac{\sqrt{2\mu(V_C(R) - Q)}}{\hbar} = 12.38\text{fm}^{-1}$$

We then have $2\kappa L = 74.75$ and $P_T = 4e^{-2\kappa L} \approx 1.38 \times 10^{-32}$. Since the tunneling probability is low, the approximation we took in considering $P_T = 4e^{-2\kappa L}$ instead of the exact expression is a good one.

e) *Besides the tunneling probability, to calculate the spontaneous fission rate we need to calculate the frequency $f = \frac{v}{R}$ for the reduced effective particle to be at the edge of the Coulomb potential. Here v is the reduced particle speed inside the nuclear well (depth $V_0 = 35$MeV) when taking Q as the (classical) energy.*

Solution:

Note: Q is the total energy of the (effective) particle. Thus the kinetic energy is $T_K = Q - V$ but $V = -V_0$ inside the well, thus we have $\frac{1}{2}\mu v^2 = Q + V_0$. Since in the problem statement I wrote *taking Q as the (classical) kinetic energy* (now corrected above) I considered correct the solution with $\frac{1}{2}\mu v^2 = Q$.
From $\frac{1}{2}\mu v^2 = Q + V_0$ we can obtain the velocity,

$$v = \sqrt{\frac{2(Q + V_0)}{\mu c^2}}c = 0.092c = 2.77 \times 10^{22}\text{fm/s}$$

Note that we need to *add* V_0 since it is the well *depth* (thus a negative energy!).
The frequency is $f = \frac{v}{R} = \frac{2.77\times10^{22}}{12.50}\ \text{s}^{-1} = 2.21 \times 10^{21}\ \text{s}^{-1}$.

f) *Finally, give the decay rate λ and the half-life for spontaneous fission of Curium 250.*

Solution:

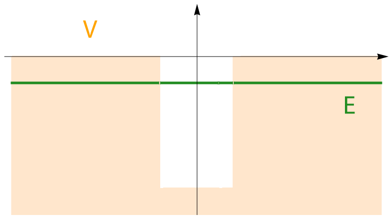
The decay rate is obtained from the same semi-classical model we studied for alpha decay. Thus it is given by the product of the frequency at which the proton is at the potential barrier (or gets separated from the parent nuclide) times the probability of tunneling through the barrier. Thus the decay rate is given by $\lambda = fP_T = 3.05 \times 10^{-11}\text{s}^{-1}$ and the half life is $t_{\frac{1}{2}} = \ln 2/\lambda = 721$years.

Problem 5: Match the potential *21 points*

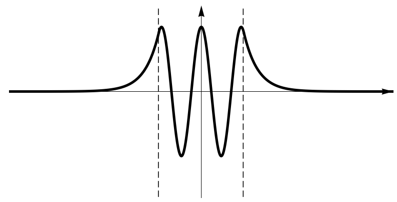
A quantum system in a 1D geometry has energy as shown by the green line and is subjected to the potential energy as in the figures labeled A-C, with 3 or 5 regions of different potential height.

*For each figure 1-7, state whether the curve plotted is a good energy eigenfunction for one (or more!) of the potentials and energies in Fig. A-C. Provide a **brief** explanation of the reasoning that lead you to your conclusion.*

(Notice: here I plot the real part of the eigenfunction).



A



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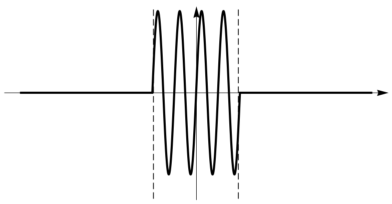

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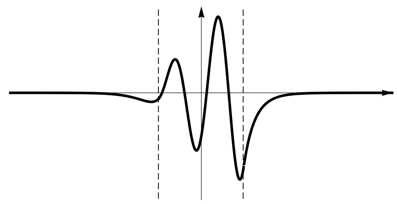

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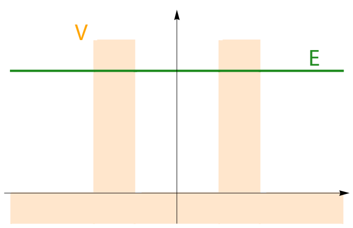
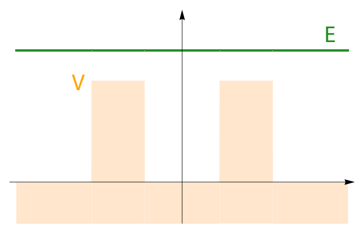
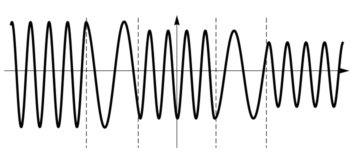


B



C



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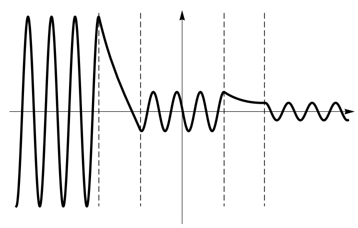

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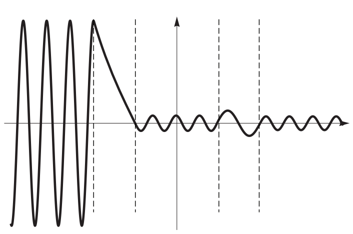

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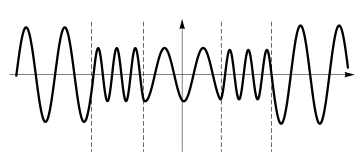

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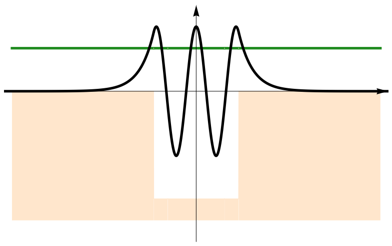
Solution:

We have the following pairings: A$\to$ 1, B$\to$ 5 and C$\to$ 4.

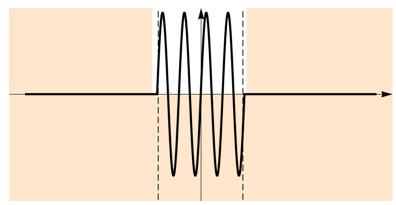
The wavefunction in 1 is a bound wavefunction with an exponential decay in the classically forbidden region, in agreement with the potential in Fig. A. Wavefunctions 2 and 3 cannot describe an eigenfunction for the potential A, 2 because it is not continuous at the boundary (while potential A has only a finite discontinuity there) and 3 because is not either an even or odd function, as required by the symmetry of the potential (it was indeed the sum of the first even solution with the first odd solution).

Wavefunction 5 represents a particle coming from the left, tunneling through a barrier, which decrease its amplitude, oscillating and then tunneling through a second barrier: this is consistent with potential B. Wavefunction 6 looks similar, but it still oscillates in region 4, so the potential should have two barriers of different height and the particle higher energy than the second barrier.

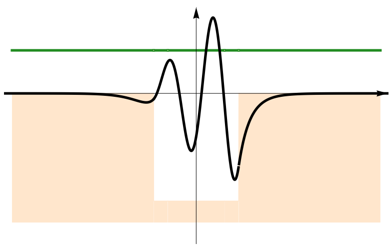
Wavefunctions 4 and 6 describe a particle with energy always higher than any potential, as in C. However, only 4 is consistent with C, while 6 would be consistent with an inverted potential (see figure), since its wavelength in the barrier region is smaller (κ larger).



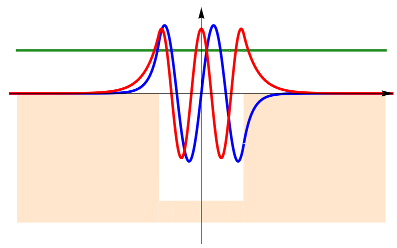
A-1



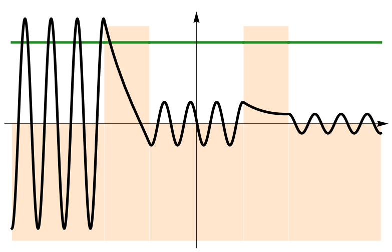
2: Infinite well



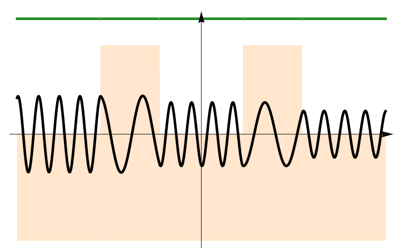
A-3: Superposition of two eigenfunctions



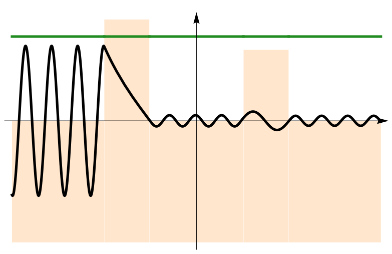
A-3: The two eigenfunctions



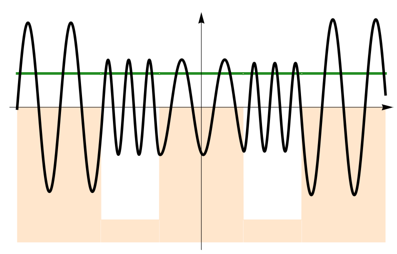
B-5



C-4



6: Unequal barriers



7: Scattering over wells (not barriers)

22.02 Introduction to Applied Nuclear Physics
Spring 2012