Lecture # 2, Quantum Computation 2: QEC Criteria

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Outline:

0. Review
1. Classical Coding
2. Q. Coding
3. Operator Measurement and Error Syndromes
4. Shor 9 Qubit Code
5. Quantum Error correction Codes Criteria (QEC criteria)

0. Review

$$\rho \xrightarrow{\varepsilon} \rho'$$

$$\varepsilon(\rho) = \sum_k E_k \rho E_k^\dagger \text{ where } \sum_k E_k E_k^\dagger = I$$

1. CLASSICAL CODING

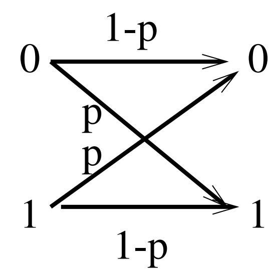


FIG. 1: a binary symmetric channel

P = prob of error

Definition: A Classical [n,k,d] code is a set of 2^k n-bit strings which have a minimum Hamming distance d.

Definition: A Hamming distance between two bit strings is $d(x, y) = w(x \oplus y)$ where $\oplus$ is the x-or operator, and w is an operation that counts the number of ones.

Example: $0_{L(ogical)} = 000$, $1_L = 111$ is a [3,1,3] code

could send	could receive	prob	decode	prob. of error
$0_L = 000$	000	$(1-p)^3$	0	
	001	$p(1-p)^2$	0	
	010	$p(1-p)^2$	0	
	100	$p(1-p)^2$	0	
	011	$p^2(1-p)$	1	$p^2(1-p)+$
	101	$p^2(1-p)$	1	$p^2(1-p)+$
	110	$p^2(1-p)$	1	$p^2(1-p)+$
	111	p^3	1	p^3

So, the total probability of error is $3p^2 - 2p^3 = O(p^2)$

2. QUANTUM CODING

1995: Thought error correction to be impossible!

1. States collapse on measurement
2. Classically error occurs or does not occur. In Q. M., errors are continuous: $\alpha|0\rangle + \beta|1\rangle \rightarrow (\alpha + \varepsilon)|0\rangle + ...$
3. No cloning Thm prohibits copying, so cannot create $\alpha|0\rangle + \beta|1\rangle \rightarrow (\alpha|0\rangle + \beta|1\rangle)(\alpha|0\rangle + \beta|1\rangle)(\alpha|0\rangle + \beta|1\rangle)$

The Solutions:

1. Measure only the effect of the environment, not the state (i.e. did an error occur?)

2\. & 3. Orthogonalize errors using entanglement: the environment has done one thing, or another, in an entangled way. $\alpha|didsomething\rangle + \beta|didnothing\rangle$

Example: The Quantum Bit Flip Code:
$|0_L\rangle = |000\rangle$
$|1_L\rangle = |111\rangle$
$|\Psi_L\rangle = \alpha|0_L\rangle + \beta|1_L\rangle$

suppose $\varepsilon(\rho) = (1 - P)\rho + PX\rho X$ where P is the probability of error and X is the error operator.

$A \stackrel{\varepsilon(\rho)}{\rightarrow} B$

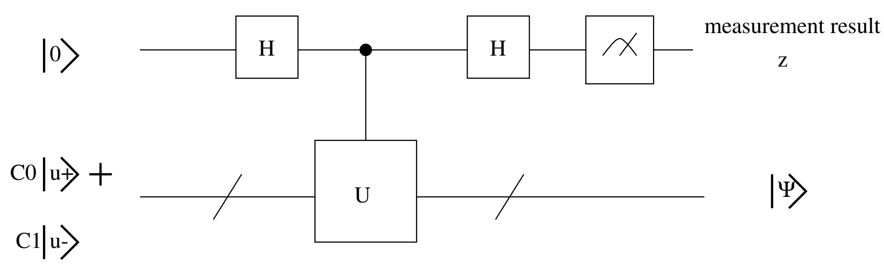
Define: An [[n,k]] quantum code C is a k-qubit subspace of an n-qubit Hilbert space. So, for our example, k=3, n=1.

Input	$\xrightarrow{\varepsilon^{\otimes 3}}$ Output	prob	decode	prob. of error
$\|\Psi\rangle = \alpha\|000\rangle + \beta\|111\rangle$	$\alpha\|000\rangle + \beta\|111\rangle$	$(1-p)^3$	0	
	$\alpha\|001\rangle + \beta\|110\rangle$	$p(1-p)^2$	0	
	$\alpha\|010\rangle + \beta\|101\rangle$	$p(1-p)^2$	0	
	$\alpha\|100\rangle + \beta\|011\rangle$	$p(1-p)^2$	0	
	$\alpha\|011\rangle + \beta\|100\rangle$	$p^2(1-p)$	1	$p^2(1-p)+$
	$\alpha\|101\rangle + \beta\|010\rangle$	$p^2(1-p)$	1	$p^2(1-p)+$
	$\alpha\|110\rangle + \beta\|001\rangle$	$p^2(1-p)$	1	$p^2(1-p)+$
	$\alpha\|111\rangle + \beta\|000\rangle$	p^3	1	p^3

3. OPERATOR MEASUREMENT

Given U with eigenvalues ± 1, eigenvectors $|u_\pm\rangle$

Definition: Measuring U



Initially, the state is $|0\rangle(C_0|u_+\rangle + C_1|u_-\rangle)$

After the first Hadamard, $\frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)(C_0|u_+\rangle + C_1|u_-\rangle)$

After the Controlled-U gate, $\frac{1}{\sqrt{2}}(|0\rangle(C_0|u_+\rangle + C_1|u_-\rangle) + |1\rangle(C_0|u_+\rangle - C_1|u_-\rangle))$

After the last Hadamard, $\frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)(C_0|u_+\rangle + C_1|u_-\rangle) + (|0\rangle - |1\rangle)(C_0|u_+\rangle - C_1|u_-\rangle)$

$= |0\rangle C_0|u_+\rangle + |1\rangle C_1|u_-\rangle$

If the measurement is $z = 0$, then $|Psi\rangle = |u_+\rangle$. (With prob $C_0^2, z = 0$)

If the measurement is $z = 1$, then $|Psi\rangle = |u_-\rangle$. (With prob $C_1^2, z = 1$)

3.1. Error Correction Syndrome Measurement

$U_1 = \sigma_z^1\sigma_z^2 = \sigma_z\sigma_z I$

$U_2 = \sigma_z^2\sigma_z^3 = I\sigma_z\sigma_z$

state	U_1	U_2
$\alpha\|000\rangle + \beta\|111\rangle$	0	0
$\alpha\|001\rangle + \beta\|110\rangle$	0	1
$\alpha\|010\rangle + \beta\|101\rangle$	1	1
$\alpha\|100\rangle + \beta\|011\rangle$	1	0

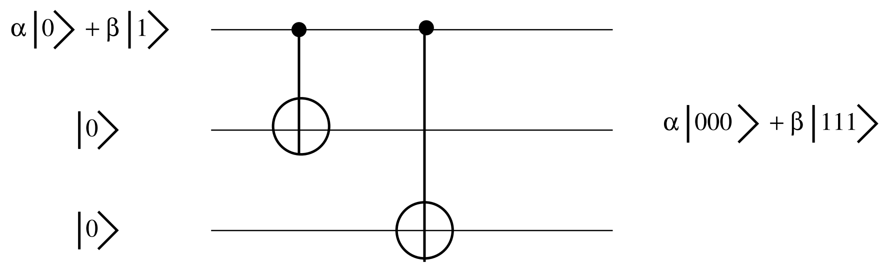
TABLE I: 0 represents a +1 eigenstate of U_i, and 1 represents a -1 eigenstate.

Steps to Error Correction:

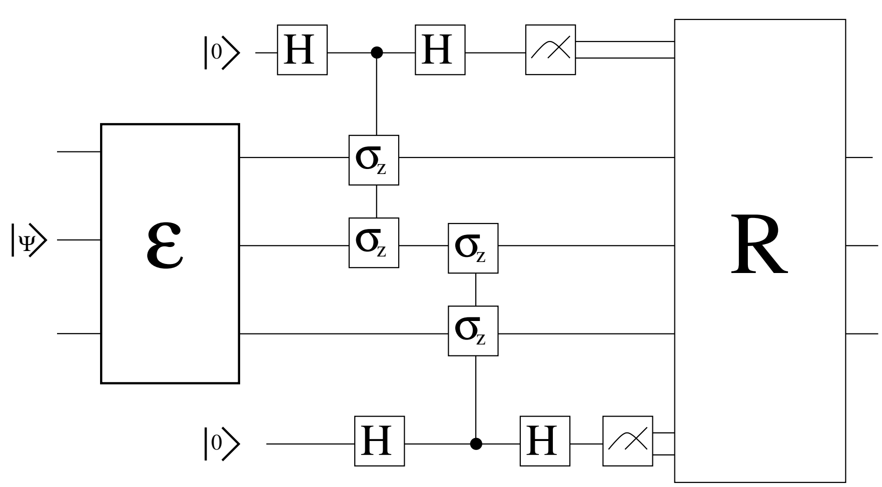
1. measure syndrome operators (here, U_1 & U_2

2. Apply recovery operator R (here, $00 \rightarrow I$, $01 \rightarrow \sigma_x^3$, $11 \rightarrow \sigma_x^2$, $10 \rightarrow \sigma_x^1$

To create the initial state $|Psi_L\rangle$:



And then to error correct:



note: the double lines indicate classical information.

Claim:

This scheme also corrects for a small continuous rotation error!

We will do this on one bit to demonstrate.

$\varepsilon(\rho) = e^{i\epsilon\sigma_x}\rho e^{i\epsilon\sigma_x}$

$e^{-\epsilon\sigma_x} = R_x(2\epsilon)$

$R_{x^1}(2\epsilon)|\Psi\rangle \cong |\Psi\rangle - i\epsilon\sigma_x^1|Psi\rangle \equiv |\Psi'\rangle$

The fidelity is $F = \sqrt{|\langle\Psi|\Psi'\rangle|^2} \cong 1 - \epsilon$

Syndrome measurement collapses error into either I or σ_x^1

$F(R(\varepsilon(\rho)), |\Psi\rangle) \cong ? \cong 1 - \epsilon^2$

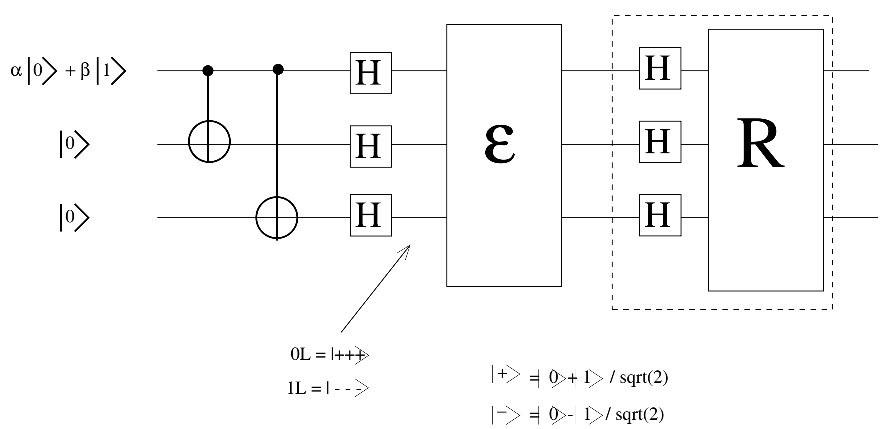
Example: The Phase Flip Code

$\varepsilon_{phaseflip}(\rho) = (1 - P)\rho + P\sigma_z\rho\sigma_z$

Recall $H\sigma_x H = \sigma_z$, $H\sigma_z H = \sigma_x$

So, $H\varepsilon_{phaseflip}(H\rho H)H = \varepsilon_{bitflip}$

Explicitly,



For the bit flip: $U_o = \sigma_z\sigma_z I$ and $U_1 = I\sigma_z\sigma_z$

For the phase flip: $U_o = \sigma_x\sigma_x I$ and $U_1 = I\sigma_x\sigma_x$

Claim:

arbitrary errors can be described as σ_x, σ_z, and $\sigma_x\sigma_z$ errors

Proof Argument:

$\varepsilon(\rho) = \sum_k E_k\rho E_k^\dagger$

where we are guaranteed $\sum_k E_k E_k^\dagger = I$

Recall pauli matrices $\sigma_j = I, \sigma_x, \sigma_y, \sigma_z$, and that $\sigma_y = -i\sigma_x\sigma_z$

Since σ_j is a basis for all 2x2 hermitian matrices, let $E_k = \sum_j C_{k_j}\sigma_j$.

Then, $\varepsilon(\rho) = \sum_{kjj'} C_{k_j} C^*_{k_{j'}} \sigma_j \rho \sigma_{j'}$

$\boxed{\varepsilon(\rho) \sum_{jj'} \chi_{jj'} \sigma_j \rho \sigma_{j'}}$ is the "Chi representation or OSR"

Example: recall

$R_x(2\epsilon)|\Psi\rangle \cong |Psi\rangle - i\epsilon\sigma_x|Psi\rangle$

$\varepsilon(\rho) = \rho - i\epsilon\sigma_x\rho - i\epsilon\rho\sigma_x + \epsilon^2\sigma_x\rho\sigma_x$

The $-i\epsilon\sigma_x\rho - i\epsilon\rho\sigma_x$ term disappears in the syndrome measurement, and the $\rho + \epsilon^2\sigma_x\rho\sigma_x$ term remains.

The result is that the syndrome measurement projects the environment into a definite error state.

4. SHOR 9 QUBIT CODE

$|0_L\rangle = (|000\rangle + |111\rangle)^{\otimes 3}/\sqrt{8}$

$|1_L\rangle = (|000\rangle - |111\rangle)^{\otimes 3}/\sqrt{8}$

this code will correct ANY single qubit error.

Syndrome Measurements:

for a bit flip: $\sigma_z^1\sigma_z^2, \sigma_z^2\sigma_z^3, \sigma_z^4\sigma_z^5, \sigma_z^5\sigma_z^6, \sigma_z^7\sigma_z^8, \sigma_z^8\sigma_z^9$

for a phase flip: $\sigma_x^1\sigma_x^2, \sigma_x^3\sigma_x^4, \sigma_x^4\sigma_x^5, \sigma_x^5\sigma_x^6, \sigma_x^6\sigma_x^7, \sigma_x^8\sigma_x^9,$

5. QEC CRITERIA/CONDITIONS

Channel: $E(\rho) = \sum_k E_k \rho E_k^\dagger$

Thm: Let C be a quantum Code defined by the orthonormal states $\{ \ |\Psi_l\rangle \ \}$

$\exists$ a quantum recovery operation R correction ε on C iff:

1. Orthogonality:

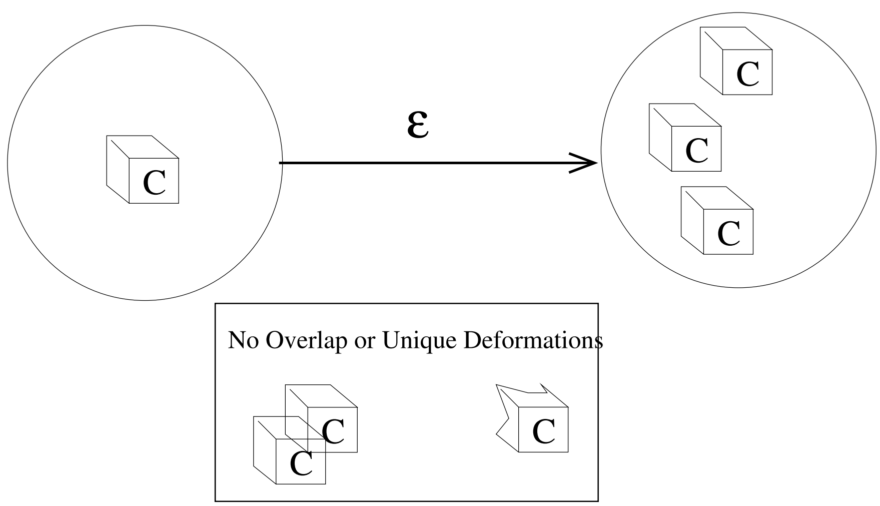
   if I have 2 errors j and k,

   $\langle\Psi_l|E_j^\dagger E_k|\Psi_l\rangle = 0$

2. Nondeformation criteria:

   $\langle\Psi_l|E_k^\dagger E_k|\Psi_l\rangle = d_k \forall l$

   this is so you cannot distinguish shrinking on different code words, all shrinking is the same.



note that d_k implies probability loss, but not information loss, $\sum_k d_k = 1$ since $\sum_k E_k^\dagger E_k = 1$

Proof: $(\rightarrow)$

Let $P = \sum_l |\Psi_l\rangle\langle\Psi_l|$ (project onto C)

note $PE_j^\dagger E_k P = d_k \delta_{jk} P$ (*)

note by Polar decomposition (extracting rotation and shrinkage) $E_k P = U_k \sqrt{PE_k^\dagger E_k P} = \sqrt{d_k} U_k P$ where $\sqrt{d_k}$ is the shrinkage and $U_k P$ is the rotation.

1. Syndrome measurement:

let $P_k = U_k P U_k^\dagger = \frac{E_k P U_k^\dagger}{\sqrt{d_k}} = \frac{U_k P E_k^\dagger}{\sqrt{d_k}}$

By (*), the P_ks are orthogonal:

$\forall k \neq j,\ P_k P_j \propto U_k P E_k^\dagger E_j P U_j^\dagger = 0$

measure P_k output k syndrome.

2. Apply Recovery R

$R(\rho) = \sum_k U_k^\dagger P_k \rho P_k U_k$

note for $|\Psi\rangle \in C$,

$U_k^\dagger P_k E_j |\Psi\rangle = \frac{U_k^\dagger U_k P E_k^\dagger}{\sqrt{d_k}} E_j P |\Psi\rangle = \frac{\delta_{jk} d_k P}{\sqrt{d_k}} |\Psi\rangle = \sqrt{d_k}\delta_{jk}|\Psi\rangle$

Thus:

$R(\varepsilon(|\Psi\rangle\langle\Psi|) = R(\sum_j E_j |\Psi\rangle\langle\Psi| E_j^T) = \sum_{jk} U_k^\dagger P_k E_j^\dagger P_k U_k = \sum_{jk} d_k \delta_{jk} P = |\Psi\rangle\langle\Psi|$