03/11/13, Eikonal Equations, Superposition of EM Waves

Lecture Note (Nick Fang)

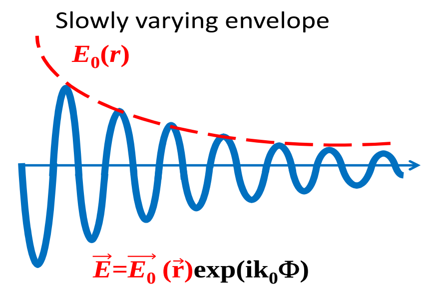
Outline:

- **Connection of EM wave to geometric optics**
- **Path of Light in an Inhomogeneous Medium**
- **Superposition of waves, coherence**

A. High Frequency Limit, connection to Geometric Optics:

How can we obtain Geometric optics picture such as ray tracing from wave equations? Now let's go back to real space and frequency domain (in an isotropic medium but with spatially varying permittivity $\varepsilon(x,z)$, for example).

$$\frac{\partial^2}{\partial x^2}E_x + \frac{\partial^2}{\partial z^2}E_x + \varepsilon(x,z)\left(\frac{\omega^2}{c_0^2}\right)E_x = 0$$

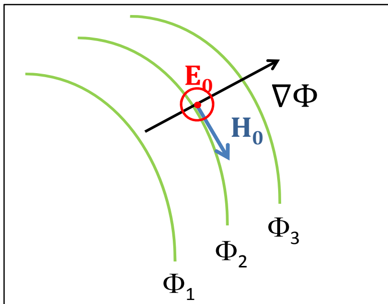


Example of decomposition of E field into the product of slowly varying envelope and a fast oscillating phase $\exp(ik_0\Phi)$

To see the connection to geometric optics, we decompose the field $\mathbf{E}(\mathbf{r}, \omega)$ into two forms: a fast oscillating component $\exp(ik_0\Phi)$, $k_0 = \omega/c_0$ and a slowly varying envelope $\mathbf{E}_0(\mathbf{r})$ as illustrated in the textbox.

Furthermore, if the envelope of field varies slowly with wavelength (e.g. $\frac{1}{k}\frac{\partial}{\partial x}E_0 \ll 1, \frac{1}{k}\frac{\partial}{\partial z}E_0 \ll 1$) then we can convert wave equations to the well-known Eikonal equation:

$$\left(\frac{\partial\Phi}{\partial x}\right)^2 + \left(\frac{\partial\Phi}{\partial z}\right)^2 = \varepsilon(x,z) = n^2(x,z)$$



Geometrical relationship of E, H, and $\nabla\Phi$

Observation (not proof):

The above equation yields: $|\nabla\Phi|^2 = n^2$, or $|\nabla\Phi| = n$.

This is equivalent to the Fermat's Principle on optical path length (OPL):

$$OPL = \int |\nabla\Phi| dl = \int n dl.$$

Such process requires the direction of the light path $\overrightarrow{dl}$, follows exactly the gradient of phase contour $\nabla\Phi$ (a vector). We will use it to determine the path of light in a general inhomogeneous medium.

B. Path of Light in an Inhomogeneous Medium

- **Example 1: 1D problems (Gradient index waveguides, Mirage Effects)**

The best known example of this kind is probably the Mirage effect in desert or near a seashore, and we heard of the explanation such as the refractive index increases with density (and hence decreases with temperature at a given altitude). With the picture in mind, now can we predict more accurately the ray path and image forming processes?

Starting from the Eikonal equation and we assume $n^2(x,z)$ is only a function of x, then we find:

$$\left(\frac{\partial\Phi}{\partial x}\right)^2 + \left(\frac{\partial\Phi}{\partial z}\right)^2 = n^2(x)$$

Since there is the index in independent of z, we may assume the slope of phase change in z direction is linear:

$$\left(\frac{\partial\Phi}{\partial z}\right) = C(const)$$

This allows us to find

$$\frac{\partial\Phi}{\partial x} = \sqrt{n^2(x) - C^2}$$

From Fermat's principle, we can visualize that direction of rays follow the gradient of phase front:

$$n\frac{d\vec{r}}{dl} = \nabla\Phi$$

z-direction: $$n(x)\frac{dz}{dl} = \mathrm{C}$$

x-direction: $$n(x)\frac{dx}{dl} = \sqrt{n^2(x) - C^2}$$

Therefore, the light path (x, z) is determined by:

$$\frac{dz}{dx} = \frac{\mathrm{C}}{\sqrt{n^2(x) - C^2}}$$

Hence

$$z - z_0 = \int_{x_0}^{x} \frac{\mathrm{C}}{\sqrt{n^2(x) - C^2}}\,dx$$

Without loss of generality, we may assume a quadratic index profile along the x direction, such as found in gradient index optical fibers or rods:

$$n^2(x) = n_0^2(1 - \alpha x^2)$$

$$z - z_0 = \int_{x_0}^{x} \frac{\mathrm{C}}{\sqrt{n_0^2(1 - \alpha x^2) - C^2}}\,dx$$

To find the integral explicitly we may take the following transformation of the variable x:

$$x = \sqrt{\frac{n_0^2 - C^2}{n_0^2\alpha}}\,sin\theta$$

Therefore,

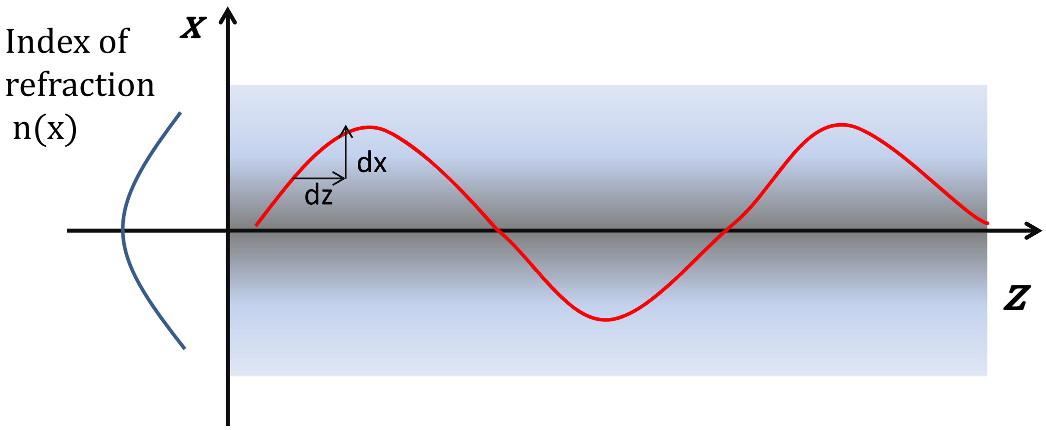
$$z - z_0 = \int_{\theta_0}^{\theta} \frac{\mathrm{C}}{n_0\sqrt{\alpha}}\,d\theta$$

$$z = z_0 + \frac{\mathrm{C}}{n_0\sqrt{\alpha}}(\theta - \theta_0)$$

Or more commonly,

$$x\sqrt{\frac{n_0^2\alpha}{n_0^2 - C^2}} = sin\theta = sin\left(\theta_0 + \sqrt{\frac{n_0^2\alpha}{C^2}}(z - z_0)\right)$$

As you can see in this example, ray propagation in the gradient index waveguide follows a sinusoid pattern! The periodicity is determined by a constant $\frac{2\pi C}{n_0\sqrt{\alpha}}$.



Observation: the constant C is related to the original "launching" angle β of the optical ray. To check that we start by:

$$\left.\frac{dz}{dx}\right|_{x=x_0} = \frac{\mathrm{C}}{\sqrt{n^2(x_0) - C^2}}$$

If we assume C=$n(x_0)cos\beta$, then

$$\left.\frac{dz}{dx}\right|_{x=x_0} = \frac{\cos\beta}{sin\beta} = cot\beta$$

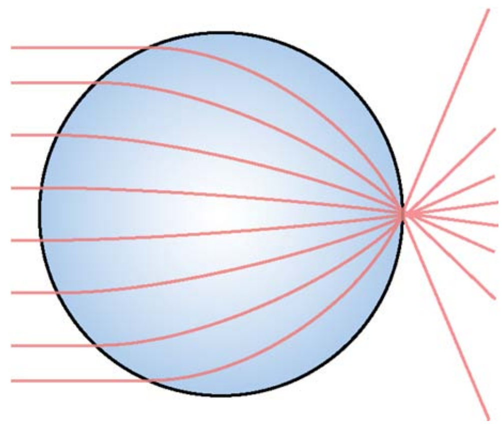
- **Other popular examples: Luneberg Lens**

  The Luneberg lens is inhomogeneous sphere that brings a collimated beam of light to a focal point at the rear surface of the sphere. For a sphere of radius R with the origin at the center, the gradient index function can be written as:

$$n(r) = \begin{cases} n_0\sqrt{2 - \frac{r^2}{R^2}}, & r \le R \\ n_0 & r > R \end{cases}$$

Such lens was mathemateically conceived during the 2nd world war by R. K. Luneberg, (see: R. K. Luneberg, Mathematical Theory of Optics (Brown University, Providence, Rhode Island, 1944), pp. 189-213.) The applications of such Luneberg lens was quickly demonstrated in microwave frequencies, and later for optical communications as well as in acoustics. Recently, such device gained new interests in in phased array communications, in illumination systems, as well as concentrators in solar energy harvesting and in imaging objectives.

Image of Luneberg lens removed due to copyright restrictions.



Left: Picture of an Optical Luneberg Lens (a glass ball 60 mm in diameter) used as spherical retro-reflector on **Meteor-3M** spacecraft. (Nasa.gov)
Right: Ray Schematics of Luneberg Lens with a radially varying index of refraction. All parallel rays (red solid curves) coming from the left-hand side of the Luneberg lens will focus to a point on the edge of the sphere.

C. Superposition of Waves

- Waves in complex numbers
  For example, the electric field of light field can be expressed as:

$$E(z,t) = Acos(kz - \omega t + \varphi)$$

Since $\exp(ix) = \cos(x) + i\sin(x)$

$$E(z,t) = Re\{Aexp[i(kz - \omega t + \varphi)]\}$$

Or

$$E(z,t) = \frac{1}{2}\{Aexp[i(kz - \omega t + \varphi)]\} + c.c. (complex\ conjugate)$$

- Complex numbers simply optics!

- Coherence