MASSACHUSETTS INSTITUTE OF TECHNOLOGY

**2.71 Optics** <span style="float:right">**Spring '09**</span>

Problem Set #4 <span style="float:right">Posted March 16, 2009 — Due <u>Wednesday, April 1, 2009</u></span>

1. **Particle in quadratic and linear potential** Consider a particle whose Hamiltonian is given by

$$H(q_x, q_z; p_x, p_z) = \frac{p_x^2 + p_y^2}{2m} + \frac{1}{2}kq_x^2 + mgq_x, \tag{1}$$

where $m$ is the particle mass, $k$ is a spring constant, and $g$ is a constant with units of acceleration. For simplicity, we have neglected the $y$ dimension, *i.e.* we have restricted the motion to take place on the $xz$ plane.

a) Write down the Hamiltonian equations as in Notes p. wk5–b–7, and reduce them to a $4 \times 4$ set of ordinary differential equations for the dynamical variables $q_x$, $q_y$, $p_x$, $p_y$ by taking the appropriate partial derivatives on (1).

b) Noting that the equations for $(q_x, p_x)$ and $(q_z, p_z)$ can be decoupled from each other, derive a single second–order differential equation for each pair and then solve these second order differential equations with initial conditions
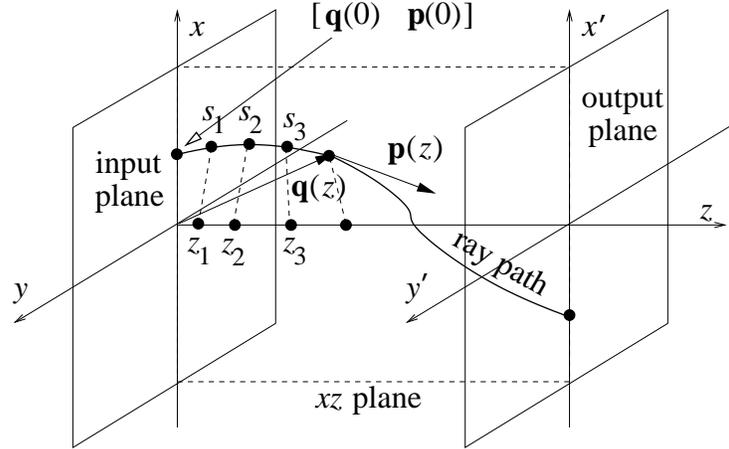
$$q_x(t = 0) = q_0, \qquad q_z(t = 0) = 0;$$
$$p_x(t = 0) = p_0, \qquad p_z(t = 0) = p_{z0}.$$

c) Give an example of a physical system that conforms to this model.

d) Is the Hamiltonian conserved in this system? <u>Hint:</u> Compute the derivative $\mathrm{d}H/\mathrm{d}t$ using (1).

2. **Quadratic GRIN** Consider the quadratic GRIN profile that is most often implemented in practice,

$$n(x) = n_0^2 - \frac{\alpha}{2}q_x^2. \tag{2}$$

Unlike the mechanical system of Problem 1, the Hamiltonian equations cannot be solved analytically in this case, so we will integrate them numerically. Before that, however, we will make some simplifications.

Consider the diagram on the next page. According to the derivation that we did in class, the position vector $\mathbf{q} = (q_x, q_y, q_z)$ and momentum vector $\mathbf{p} = (p_x, p_y, p_z)$ are parameterized by the variable $s$. The "time" variable $s$ indexes location along the ray in the sense that as we advance $s$ from 0 to $s_1$ to $s_2$ to $s_3$ and so on, we progress along successive locations on the ray path (see the figure.)

It turns out that we can eliminate the variable $s$ and instead parameterize with respect to a spatial variable, say $z$. Graphically, this is also shown in the figure, where $s_1, s_2, \ldots$ have been projected to their corresponding $z_1, z_2, \ldots$ representing locations on the optical axis. Now the ray is parameterized as $[\mathbf{q}(z),\ \mathbf{p}(z)]$ instead of $[\mathbf{q}(s),\ \mathbf{p}(s)]$. It can be shown that the ray trajectory satisfies a new set of Hamiltonian equations (in terms of $z$)

$$\frac{\mathrm{d}q_x}{\mathrm{d}z} = \frac{\partial h}{\partial p_x}, \qquad \frac{\mathrm{d}p_x}{\mathrm{d}z} = -\frac{\partial h}{\partial q_x},$$
$$\frac{\mathrm{d}q_y}{\mathrm{d}z} = \frac{\partial h}{\partial p_y}, \qquad \frac{\mathrm{d}p_y}{\mathrm{d}z} = -\frac{\partial h}{\partial q_y}.$$

The quantity $h$ is referred to as "Screen Hamiltonian," and is given by

$$h\left(q_x, q_y, z; p_x, p_y\right) = -\sqrt{n^2\left(\mathbf{q}\right) - \left(p_x^2 + p_y^2\right)}.$$

We can make sense of the Screen Hamiltonian as follows: if we slice the ray trajectories with successive planes ("screens") oriented perpendicular to the optical axis $z$, then the set of Screen Hamiltonian equations describes the evolution of the *intersection* of the ray path with the screens as $z$ advances. That is, we can solve the Screen Hamiltonian equations starting with the initial ray location $(q_x, q_y)$ and momentum (angle of propagation) $(q_x, q_y)$ at an "input plane" $z = 0$ and solve for the location and angle at any "output plane" located at $z > 0$ (see again the figure above.) The proof of why this is so is given in detail in Problem Set #4 of 2.710, but we will skip it here. Instead, we will solve the Screen Hamiltonian numerically to get a feel for how it works with the quadratic GRIN.

Download from the class website the Matlab functions `bundle_raytrace.m`, `screen_raytrace.m`, `sgradh_elliptical.m`, and `sgradh_quadratic_hw.m`. The main function `screen _raytrace` is the solver that computes ray trajectories numerically from the set of Screen Hamiltonian equations , while `bundle_raytrace` calls the solver repeatedly for a collimated ray bundle parallel to the optical axis $z$ and several values of $q_0$.

Note that the Matlab functions are for two–dimensional screens $\mathbf{q} = (q_x, q_y)$, $\mathbf{p} = (p_x, p_y)$ but you can ignore the 2nd ($y$) dimension by setting all arguments equal to zero.

    a) Call `bundle_raytrace` with `sgradh_elliptical` as argument to numerically ray–trace the elliptical GRIN $n(x) = \sqrt{n_0^2 - \kappa^2 q_x^2}$. (This GRIN actually admits an analytical solution, but we're not asking you to find it.) Make sure that you study and understand the structure of `sgradh_elliptical.m`. Ray trace the case $n_0 = 1.5$, $\kappa = 0.1$ for initial values $q(z = 0) \equiv q_0$ ranging from $-10 : 10$ and for $0 \leq z \leq 25$.

    b) In your opinion, does the elliptical GRIN focus the incident parallel ray bundle satisfactorily? What is the dominant aberration that you observe?

    c) Now consider `sgradh_quadratic_hw.m` for solving the quadratic GRIN (2). Edit the function to enter the proper value of the derivative $\mathrm{d}p_x/\mathrm{d}z$. Ray trace again by calling `bundle_raytrace` with `sgradh_quadratic` as argument. How does the performance of the quadratic GRIN compare with the elliptical GRIN in terms of focusing quality?

    d) Check your numerical results to verify that the Screen Hamiltonian is conserved in your solution. This is true here because the index of refraction does not depend on the parameterization variable $z$. In the general case $\partial n/\partial z \neq 0$, the Screen Hamiltonian is not conserved! This is one important difference compared to the regular Hamiltonian.

3. **Complex arithmetic** In Wave Optics, the use of complex numbers, in particular phasors, is prevalent because it considerably simplifies calculations of interference and diffraction. The goal of this exercise is to remind you of some basic complex arithmetic. Let $z_1 = 3 + i4$, $z_2 = 1 - i$, $z_3 = 5e^{i\pi/3}$, and $z_4 = 5e^{i4\pi/3}$. Compute, in the easiest way possible, and without use of electronic calculators, the following quantities:

    **3.a)** $|z_1|, |z_2|, |z_3|, |z_4|, \angle z_1, \angle z_2, \angle z_3, \angle z_4$;

    **3.b)** $\angle - z_1, \angle z_2{}^*, \angle - z_3, \angle z_4{}^*$; ($z^*$ denotes the complex conjugate of the complex number $z$;)

    **3.c)** $z_1 + z_2, z_1{}^* + z_2, z_3 + z_4, z_1 - z_4{}^*$;

    **3.d)** the magnitudes and phases of $z_1 z_2, z_3 z_4, z_3/z_4, \sqrt{z_3}$;

    **3.e)** $z_1 + e^{i\pi}, e^{i\pi/2} z_2, e^{i\pi} z_3, \sqrt{e^{-i\pi} z_4}$.

4. **Plane waves and phasor representations** Throughout this problem, by "real expression" of a wave we mean the space-time representation, e.g. $f(x, y, z, t) = A\cos(kz - \omega t)$ is a plane wave of wave-vector magnitude $k$ and frequency $\omega$

propagating in the direction of the $\hat{\mathbf{z}}$ coordinate axis. By "phasor expression" we mean the complex representation of the wave, e.g. $Ae^{ikz}$ for the same wave.

**4.a)** Write the real and phasor expressions for a plane wave $f_1(x, y, z, t)$ propagating at an angle $30°$ relative to the $\hat{\mathbf{z}}$ axis on the $xz$-plane (*i.e.*, the plane $y = 0$). The wavelength is $\lambda = 1\mu\text{m}$, and the wave speed is $c = 3 \times 10^8 \text{m} \cdot \text{sec}^{-1}$.

**4.b)** Write the real and phasor expressions for a plane wave $f_2(x, y, z, t)$ of the same wavelength and wave speed as $f_1$ but propagating at angle $60°$ relative to the $\hat{\mathbf{z}}$ axis on the $yz$-plane.

**4.c)** Use the real expression to plot $f_1(x, y, z = 0, t = 0)$ and $f_2(x, y, z = 0, t = 0)$ using MATLAB. (*Note:* you will probably need to use `surf` or an equivalent command.)

**4.d)** The plane $z = 0$ is illuminated by the superposition of the two waves $f_1$ and $f_2$. Plot the waveform received at points A, B, C, D, E with Cartesian coordinates, respectively,

$$(0, 0, 0), \ \left(\frac{1}{4}, -\frac{1}{4\sqrt{3}}, 0\right), \ \left(\frac{1}{2}, -\frac{1}{2\sqrt{3}}, 0\right), \ \left(\frac{3}{4}, -\frac{3}{4\sqrt{3}}, 0\right), \ \left(1, -\frac{1}{2\sqrt{3}}, 0\right)$$

(all units in microns) as function of time. What do you observe?

5. **Wave superposition** Consider the following two waves,

$$f_1(x, z, t) = 5\cos\left(\frac{2\pi}{17}\left[z + \frac{x^2}{2z}\right] - 2\pi 10t\right)$$

$$f_2(x, z, t) = 5\cos\left(\frac{2\pi}{17}\left[z + \frac{(x-5)^2}{2z}\right] - 2\pi 10t + \frac{\pi}{3}\right)$$

**a)** What is the physical interpretation of these waves? Be as detailed as possible.

**b)** If these waves approximately satisfy the (Helmholtz) Wave Equation, what is the phase velocity?

**c)** Using trigonometric identities, calculate the wave that results from the "coherent superposition" of the two waves, *i.e.* $f(x, z, t) \equiv f_1(x, z, t) + f_2(x, z, t)$.

**d)** Now express each wave as phasor, add the two phasors and compare the result to the phasor of $f(x, z, t)$ from part (b).

4

6. **Dispersive waves** Consider the metallic waveguide from Notes p. wk7–a–1.

   a) An electromagnetic wave of frequency $\omega = 1.5 \times 10^{15}$rad/sec is launched into the waveguide such that at $t = 0$ the amplitude is maximum at a distance $0.4\mu$m inside the guide. Plot the wave 1.05 fsec, 2.1 fsec, and 3.15 fsec later.

   b) Compare the distance traveled by a point of constant phase on the wavefront in 4.2 fsec to the distance traveled in the same time by a point of constant phase on the wavefront of a wave of the same frequency but propagating in free space.

   c) Show that the group velocity is

   $$v_{\mathrm{g}} = c\sqrt{1 - \left(\frac{m\pi c}{a\omega}\right)^2}.$$

7. **Schroedinger's Equation** The equation describing the wavepacket associated with a particle in Quantum Mechanics is

   $$\frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2} + \frac{\partial^2 \psi}{\partial z^2} = -i\frac{2m}{\hbar}\frac{\partial \psi}{\partial t},$$

   where $m$ is the particle mass, $\hbar = h/2\pi$ and $h$ is Planck's constant.

   a) Interpret the $i$ appearing in this equation.

   b) Use a procedure similar to Notes p. wk6–b–16 to derive the dispersion relation for Schroedinger waves and plot it as a dispersion diagram.

2.71 / 2.710 Optics
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