

Lecture #4-5 Instructor Notes

Lets's start off ray theory with a little history. As ray theory is obviously very, very well known to all students at this level, maybe a little sports history analogy/reference would be amusing. My favorite is by legendary Green Bay Packers coach Vince Lombardi, who sarcastically told his players after a loss.....

"Gentlemen, this is a football." - Vince Lombardi



Photo courtesy of [kenfagerdotcom](#) on Flickr. CC-BY-NC-SA.

Max McGee said....."Coach, you're going a little too fast."

This is about the same for Snell's Law and graduate students. .

However, as bright as all grad students are, I find that far too few take the bit of extra time and effort needed to learn the history of their craft. To show you how this can be interesting in itself, let me include a few tidbits about the "well known" Snell's Law!

Please read the following [Wikipedia](#) pages.

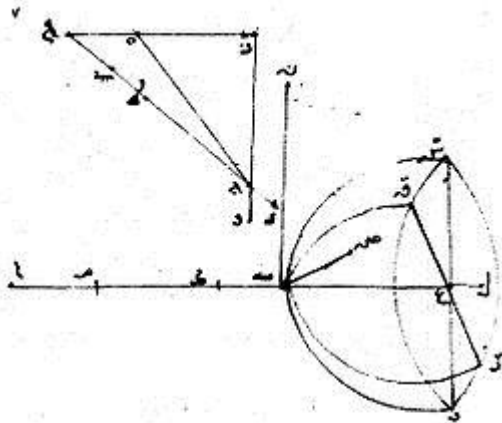
[Willebrord Snellius \(born Willebrord Snel van Royen\), 1580-1626](#)



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The most famous law in optics (and acoustics, and gravitational lensing, etc) is mis-attributed! (As are many others in science and engineering....) Let's see who really first described Snell's Law!

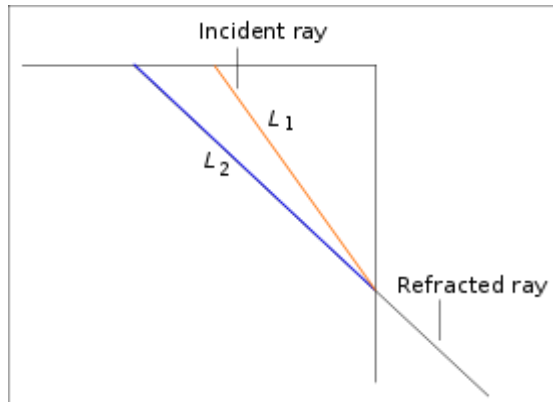
[Ibn Sahl, 940-1000](#)



لانه ان ماتة عليها سطح مستوي غيره فلا ان هذا السطح يتقطع سطح برتر
على نقطة تب تلا بد من ان يتقطع احد خطي م ب ن بص فليكن ذلك
الخط مستوي والفصل المتكافئ بين هذا السطح وبين سطح قطع و ر
خط مستوي فلا ان هذا السطح ياتر سطح م ب على نقطة تب بخط
م ب تخط تر قطع و ر م ب على نقطة تب وكذلك خط مستوي غير
فلا ياتر سطح م ب على نقطة تب سطح مستوي غير سطح م ب ن ص

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Reproduction of a page of Ibn Sahl's manuscript showing his discovery of the law of refraction (from Rashed, 1990).



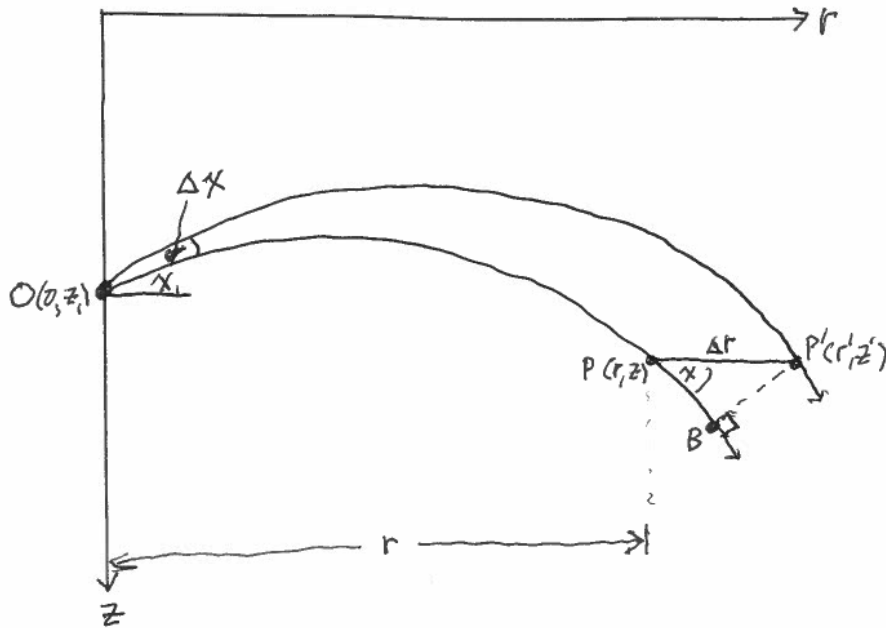
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Interpretation of Ibn Sahl's construction. If the ratio of lengths L_1 / L_2 is kept equal to n_1 / n_2 then the rays satisfy the law of sines, or Snell's law.

Ray Theory Intensity (Transport Equation)

Ray theory does a very good job with phases. Amplitudes/intensities are not exactly it's strong suit, however. But if one is content to live with approximate intensities, and ignore some pathologies (caustics and shadow zones), then ray theory intensity calculations can be useful, as well as giving a lot of physical insight. Let's go through a rather basic, simple derivation of acoustic intensity using ray theory.

Consider from the figure below the sound field due to an omnidirectional point source at $O(0, z_1)$. We want to find the sound intensity I (energy flux per unit time through a unit cross sectional area) at some arbitrary point $P(r, z)$. To do this, we consider the spreading of an elementary ray tube, depicted below.



From the figure,

$$r = r(\chi_1)$$

$$PP' = \Delta r = \left| \frac{\partial r}{\partial \chi_1} \right| \Delta \chi_1$$

The cross section of the ray tube in the plane of the drawing is thus:

$$BP' = PP' \sin \chi$$

If we have azimuthal symmetry (e.g. a layered medium in z), the spread of the ray tube in the plane $z = \text{const}$ is $\Delta \phi$. Then the area of the transverse cross section of the tube is:

$$dS = \left| \frac{\partial r}{\partial \chi_1} \right| \Delta \chi_1 r \sin \chi \Delta \phi$$

To get the power radiated into the ray tube, we multiply the total radiated power W by the ratio of the solid angle the ray tube subtends at $r=1$ to the 4π solid angle of the unit sphere. This gives:

$$dW = \frac{1}{4\pi} W \cos \chi_1 \Delta \chi_1 \Delta \phi$$

The energy flux (watts/m²) or sound intensity is given by

$$I = dW / dS = \frac{W}{4\pi r} \frac{\cos \chi_1}{\sin \chi \left| \frac{\partial r}{\partial \chi_1} \right|}$$

We know all the above numbers from the raypath equations, so we can get intensity. If we had a homogeneous medium, we would have

$$I_0 = \frac{W}{4\pi R^2}$$

where

$$R^2 = r^2 + (z - z_1)^2$$

We define the ratio $f = I / I_0$ (inhomogeneous to homogeneous result) as the “focusing factor.” This can be written, using our previous work, as;

$$f = \frac{[r^2 + (z - z_1)^2] \cos \chi_1}{r |\partial r / \partial \chi_1| \sin \chi}$$

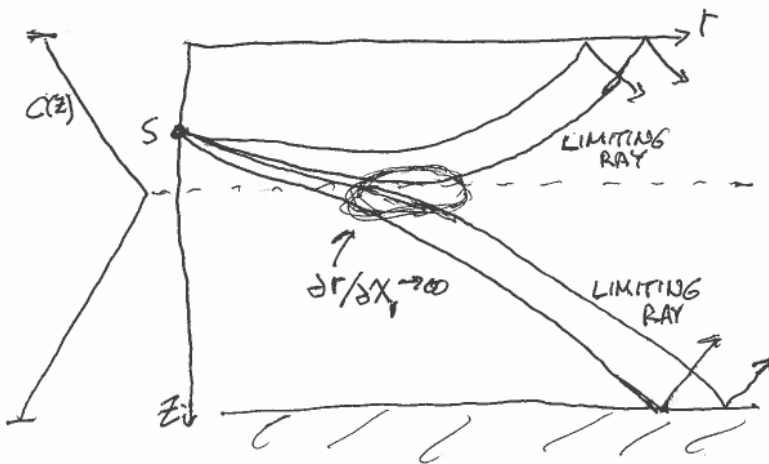
As we know, the equation for an eigenray can be written parametrically as

$$r = r(\chi_1, z)$$

If $\partial r / \partial \chi_1 = 0$, then we get the focusing factor $f \rightarrow \infty$!!! This is called a caustic, and is one of the pathologies of ray theory. Foci are a special case of caustics. If you would like to see a very nice case of a focus in ray acoustics, consider the soundspeed profile

$$c(z) = c_0 \cosh(bz)$$

We can also have a soundspeed profile which spreads (“anti-ducts”) the sound, as shown below

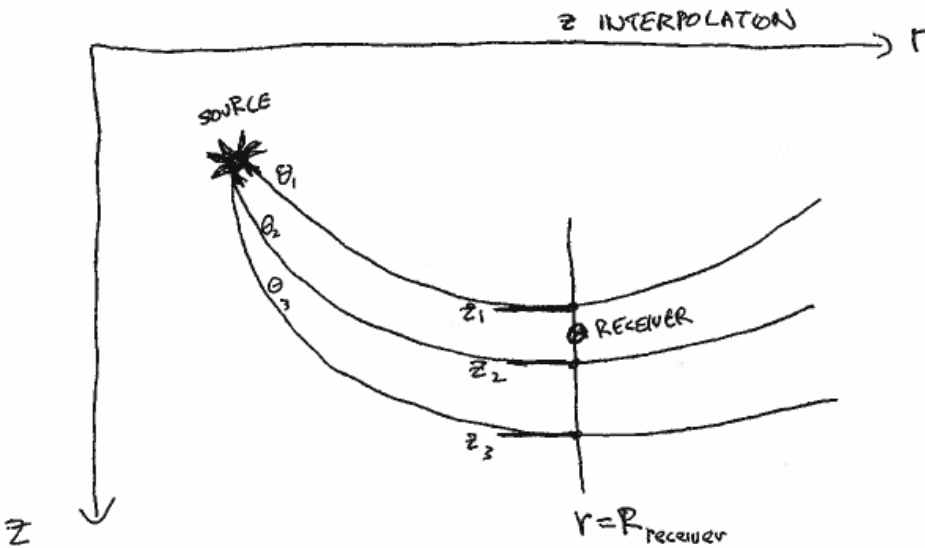


Here, $\partial r / \partial \chi_1 \rightarrow \infty$, so that $l=0$. Due to diffraction, l is not zero in this “shadow zone” region, so this is another ray theory pathology.

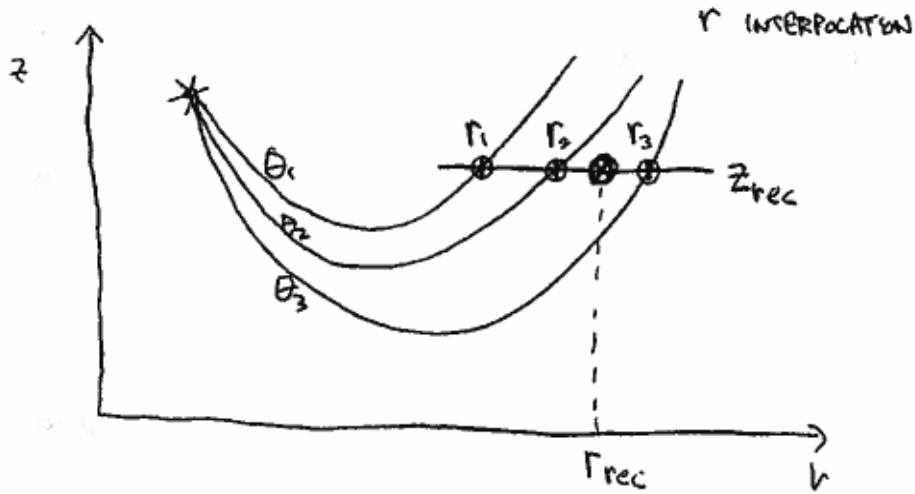
We would note that caustics and shadow zones generally go together. If one focuses light/sound in one area, it is due to reducing the level in another area.

Eigenrays

If one is doing experiments, the interesting rays are those that connect source and receiver, which in ocean acoustics are called (for reasons unknown) “eigenrays.” To find them is a straightforward exercise, performed as follows (see figure below). First, one sends out a fan of closely spaced rays at angles $\theta_1, \theta_2, \theta_3$, etc. and then sees at which z (depth) they cross the range of the receiver. We can see in the figure below that the two depths z_1 and z_2 bracket the receiver depth at that range, with z_{rec} about halfway between the two. Even though the angle and depth are not linearly related, we can still take a first guess that $\theta_{eig} \approx (\theta_1 + \theta_2) / 2$. In practice, one keeps iterating this process with a finer and finer spray of rays until one comes within a specified distance of the receiver depth.

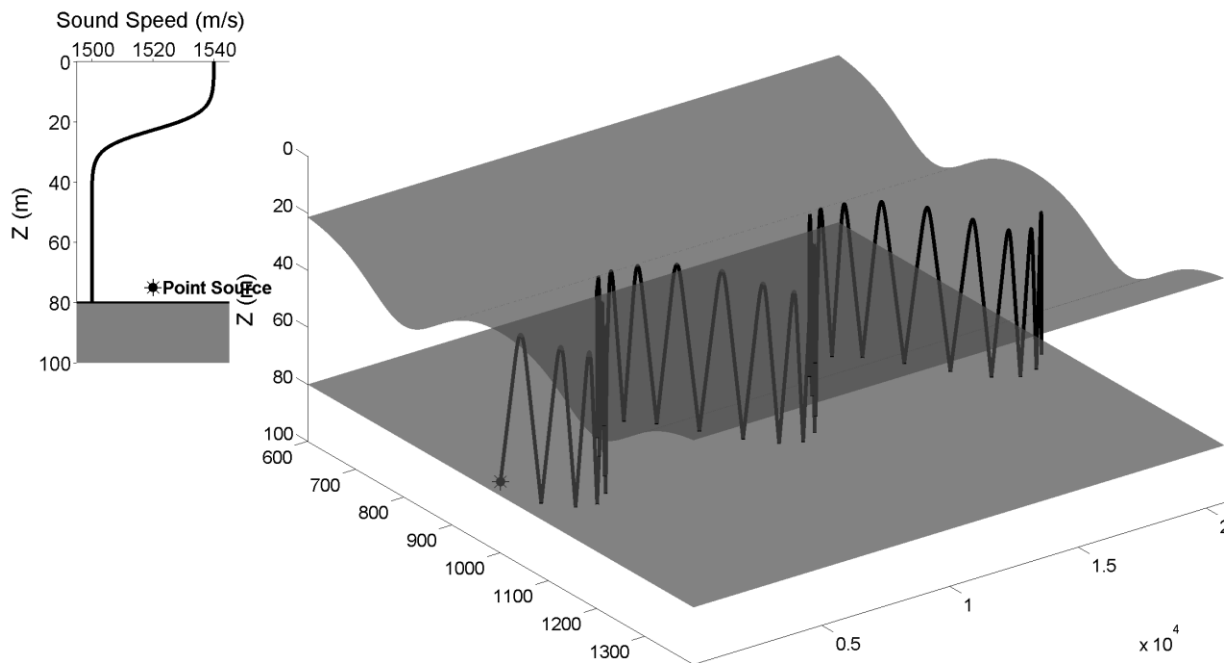


Alternately, one can interpolate in range, as seen below. In this example, we see that r_{rec} is about halfway between r_2 and r_3 , so that $\theta_{eig} \approx (\theta_2 + \theta_3) / 2$. Again, the search can be iterated and refined.



One can also use r and θ as the variables we interpolate. We will not show that here. In general, the z interpolation is most common, but the choice is there for the user, based usually on the most convenient geometry.

Ray tracing can also be done in three dimensions, if one has a situation where there is a strong enough azimuthal gradient in soundspeed or in bottom topography to deflect the sound appreciably out of the source-receiver plane. An example is shown below of a three dimensional ray path wending between two internal wave crests.



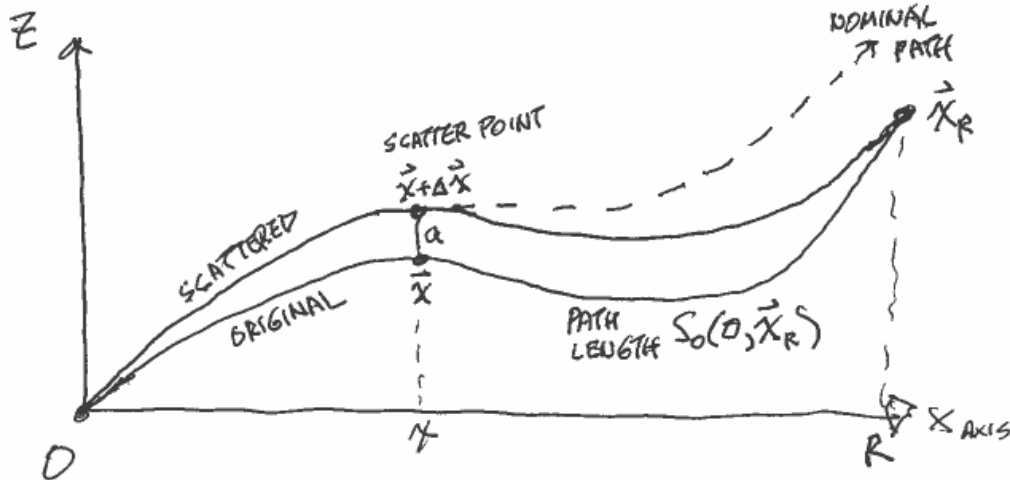
3D ducting of sound by an internal wave.

Courtesy of Y. T. Lin. Used with permission.

More complicated Fresnel Zone Tubes

In the readings, we saw a simple discussion of how a “finite frequency” ray has a finite lateral extent, defined by the Fresnel Zone or “ray tube.” This was examined for an infinite, constant soundspeed medium, which emphatically is NOT the case for the ocean. To deal with the refraction of the raypaths by the ocean, one needs to use a little bit more complicated picture, which is shown in the figure below. (This is a pseudo-2D geometry, which suffices for most cases.)

The Fresnel Zone size is defined as $\Delta\vec{x}$ (see the diagram) such that the scattered path deviates from the unscattered path by half a wavelength (i.e. π , as before). We choose $\Delta\vec{x} = (0,0,a)$. For a small, we get the difference in path length:



Fresnel zone tube for a complicated ocean sound speed profile and refracted rays.

$$S_0(0, \vec{x} + \Delta\vec{x}) + S_0(\vec{x} + \Delta\vec{x}, \vec{x}_R) - S_0(0, \vec{x}_R) = \frac{a^2}{2} \partial_{zz} [S_0(0, \vec{x}) + S_0(\vec{x}, \vec{x}_R)].$$

We note that the first order term must be zero by Fermat’s Principle, that the first variation of the raypath is zero. The second derivative of the term in square brackets is called the “phase curvature” (since second derivatives are a measure of curvature) and is denoted by $A(x)$. $A(x)$ is a measure of the rate at which the path length changes as one moves away from a given raypath.

By the definition of the Fresnel Zone (see the readings), we get that:

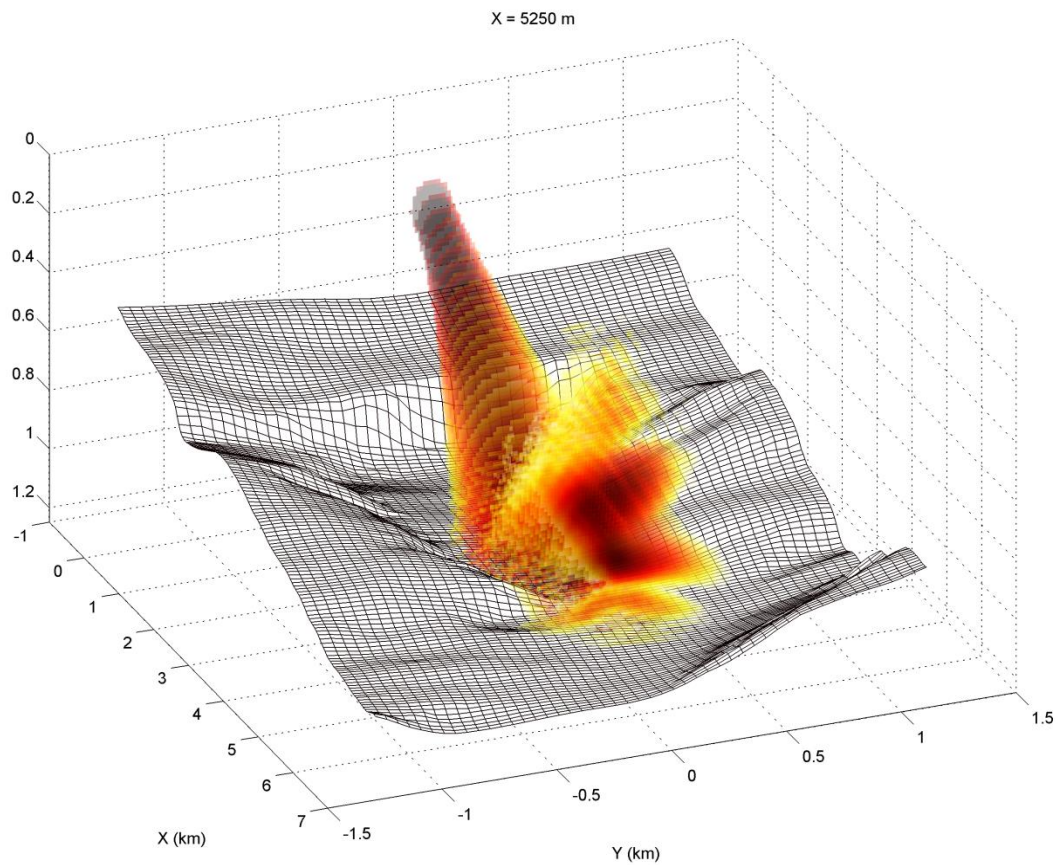
$$AR_F^2 / 2 = \lambda / 2$$

Or simply

$$R_F^2(x) = \lambda A^{-1}(x)$$

In our simple readings example, $A^{-1}(x) = h$.

As a final example of rays, let us consider a Gaussian beam of energy hitting a rough bottom. This example, shown below, demonstrates how a single beam can break into many upon reflection, and thus produce extra rays that would in turn need to be tracked.



Parabolic equation model calculation of a 3-D Gaussian beam being scattered by rough bottom topography.

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2.682 Acoustical Oceanography
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