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2.29 NUMERICAL FLUID MECHANICS — SPRING 2015

EQUATION SHEET – Quiz 1

Number Representation

- Floating Number Representation: $x = m\, b^e$, $b^{-1} \le m < b^0$

Truncation Errors and Error Analysis $y = f(x_1, x_2, x_3, \ldots, x_n)$

- Taylor Series:

$$f(x_{i+1}) = f(x_i) + \Delta x\, f'(x_i) + \frac{\Delta x^2}{2!} f''(x_i) + \frac{\Delta x^3}{3!} f'''(x_i) + \ldots + \frac{\Delta x^n}{n!} f^n(x_i) + R_n$$

$$R_n = \frac{\Delta x^{n+1}}{n+1!} f^{(n+1)}(\xi)$$

- The Differential Error (general error propagation) Formula: $\varepsilon_y \le \sum_{i=1}^{n} \left| \frac{\partial f(x_1, \ldots, x_n)}{\partial x_i} \right| \varepsilon_i$
- The Standard Error (statistical formula): $E(\Delta_s y) \simeq \sqrt{\sum_{i=1}^{n} \left(\frac{\partial f}{\partial x_i} \right)^2 \varepsilon_i^2}$
- Condition Number of *f(x)*: $K_p = \left| \frac{\bar{x}\, f'(\bar{x})}{f(\bar{x})} \right|$

Roots of nonlinear equations ($x_{n+1} = x_n - h(x_n) f(x_n)$ **)**

- Bisection: successive division of bracket in half, next bracket based on sign of $f(x_1^{n+1})\, f(x_{\text{mid-point}}^{n+1})$
- False-Position (Regula Falsi): $x_r = x_U - \frac{f(x_U)\,(x_L - x_U)}{f(x_L) - f(x_U)}$
- Fixed Point Iteration (General Method or Picard Iteration): $x_{n+1} = g(x_n)$ or $x_{n+1} = x_n - h(x_n) f(x_n)$
- Newton Raphson: $x_{n+1} = x_n - \frac{1}{f'(x_n)} f(x_n)$
- Secant Method: $x_{n+1} = x_n - \frac{(x_n - x_{n-1})}{f(x_n) - f(x_{n-1})} f(x_n)$
- Order of convergence *p*: Defining $e_n = x_n - x^e$, the order of convergence *p* exists if there exist a constant C≠0 such that: $\lim_{n\to\infty} \frac{|e_{n+1}|}{|e_n|^p} = C$

Conservation Law for a scalar ϕ, in integral and differential forms:

- $$\left\{\frac{d}{dt}\int_{CM}\rho\phi dV=\right\}\quad \frac{d}{dt}\int_{CV_{\text{fixed}}}\rho\phi dV+\underbrace{\int_{CS}\rho\phi\,(\vec{v}.\vec{n})dA}_{\substack{\text{Advective fluxes}\\ \text{(Adv.\& diff. ="convection" fluxes)}}}=\underbrace{-\int_{CS}\vec{q}_{\phi}.\vec{n}\,dA}_{\substack{\text{Other transports}\\ \text{(diffusion, etc)}}}+\underbrace{\sum\int_{CV_{\text{fixed}}}s_{\phi}\,dV}_{\substack{\text{Sum of sources and}\\ \text{sinks terms (reactions, etc)}}}$$
- $$\frac{\partial\rho\phi}{\partial t}+\nabla.(\rho\phi\vec{v})=-\nabla.\vec{q}_{\phi}+s_{\phi}$$

Linear Algebraic Systems:

- Gauss Elimination: reduction, $m_{ik}=\dfrac{a_{ik}^{(k)}}{a_{kk}^{(k)}}$, $a_{ij}^{(k+1)}=a_{ij}^{(k)}-m_{ik}\,a_{kj}^{(k)}$, $b_i^{(k+1)}=b_i^{(k)}-m_{ik}\,b_k^{(k)}$,

  followed by a back-substitution. $x_k=\left(b_k-\sum_{j=k+1}^{n}a_{kj}^{(k)}x_j\right)\Big/a_{kk}^{(k)}$
- LU decomposition: **A**=**LU**, $a_{ij}=\sum_{k=1}^{\min(i,j)}m_{ik}\,a_{kj}^{(k)}$
- Choleski Factorization: **A**=**R*****R**, where **R** is upper triangular and **R*** its conjugate transpose.
- Condition number of a linear algebraic system: $K(\mathbf{A})=\left\|\mathbf{A}^{-1}\right\|\,\left\|\mathbf{A}\right\|$
- A banded matrix of p super-diagonals and q sub-diagonals has a bandwidth $w=p+q+1$
- $\sum_{i=1}^{n}i=\dfrac{n(n+1)}{2}$ and $\sum_{i=1}^{n}i^2=\dfrac{n(n+1)(2n+1)}{6}$
- Eigendecomposition: $\mathbf{Ax}=\lambda\mathbf{x}$ and $\det(\mathbf{A}-\lambda\mathbf{I})=0$
- Norms:

$$\left\|\mathbf{A}\right\|_1=\max_{1\le j\le n}\sum_{i=1}^{m}\left|a_{ij}\right| \qquad \text{“Maximum Column Sum”}$$

$$\left\|\mathbf{A}\right\|_\infty=\max_{1\le i\le m}\sum_{j=1}^{n}\left|a_{ij}\right| \qquad \text{“Maximum Row Sum”}$$

$$\left\|\mathbf{A}\right\|_F=\sqrt{\left(\sum_{i=1}^{m}\sum_{j=1}^{n}\left|a_{ij}\right|^2\right)} \qquad \text{“Frobenius norm” (or “Euclidean norm”)}$$

$$\left\|\mathbf{A}\right\|_2=\sqrt{\lambda_{\max}\left\{\mathbf{A}^*\mathbf{A}\right\}} \qquad \text{“}L-2\text{ norm” (or “spectral norm”)}$$

Iterative Methods for solving linear algebraic systems: $\mathbf{x}^{k+1}=\mathbf{B}\,\mathbf{x}^k+\mathbf{c}\qquad k=0,1,2,...$

- Necessary and sufficient condition for convergence:

  $\rho(\mathbf{B})=\max_{i=1...n}\left|\lambda_i\right|<1$, where $\lambda_i=\text{eigenvalue}(\mathbf{B}_{n\times n})$
- Jacobi's method: $\mathbf{x}^{k+1}=-\mathbf{D}^{-1}(\mathbf{L}+\mathbf{U})\,\mathbf{x}^k+\mathbf{D}^{-1}\mathbf{b}$
- Gauss-Seidel method: $\mathbf{x}^{k+1}=-(\mathbf{D}+\mathbf{L})^{-1}\mathbf{U}\,\mathbf{x}^k+(\mathbf{D}+\mathbf{L})^{-1}\mathbf{b}$
- SOR Method: $\mathbf{x}^{k+1}=(\mathbf{D}+\omega\mathbf{L})^{-1}[-\omega\mathbf{U}+(1-\omega)\mathbf{D}]\mathbf{x}^k+\omega(\mathbf{D}+\omega\mathbf{L})^{-1}\mathbf{b}$
- Steepest Descent Gradient Method: $\mathbf{x}_{i+1}=\mathbf{x}_i+\dfrac{\mathbf{r}_i^T\mathbf{r}_i}{\mathbf{r}_i^T\mathbf{A}\mathbf{r}_i}\mathbf{r}_i$, $\mathbf{r}_i=\mathbf{b}-\mathbf{A}\mathbf{x}_i$
- Conjugate Gradient: $\mathbf{x}_{i+1}=\mathbf{x}_i+\alpha_i\,\mathbf{v}_i$ (α_i such that each $\mathbf{v}_i$ are generated by orthogonalization of residuum vectors and such that search directions are **A**-conjugate).

Finite Differences – PDE types (2nd order, 2D): $A\phi_{xx} + B\phi_{xy} + C\phi_{yy} = F(x, y, \phi, \phi_x, \phi_y)$

$B^2 - AC > 0$: hyperbolic; $B^2 - AC = 0$: parabolic; $B^2 - AC < 0$: elliptic

Finite Differences – Error Types and Discretization Properties ($\mathcal{L}(\phi) = 0$, $\hat{\mathcal{L}}_{\Delta x}(\hat{\phi}) = 0$)

- Consistency: $\left|\mathcal{L}(\phi) - \hat{\mathcal{L}}_{\Delta x}(\phi)\right| \to 0$ when $\Delta x \to 0$
- Truncation error: $\tau_{\Delta x} = \mathcal{L}(\phi) - \hat{\mathcal{L}}_{\Delta x}(\phi) \to O(\Delta x^p)$ for $\Delta x \to 0$
- Error equation: $\tau_{\Delta x} = \mathcal{L}(\phi) - \hat{\mathcal{L}}_{\Delta x}(\hat{\phi} + \varepsilon) = -\hat{\mathcal{L}}_{\Delta x}(\varepsilon)$ (for linear systems)
- Stability: $\left\|\hat{\mathcal{L}}_{\Delta x}^{-1}\right\| <$ Const. (for linear systems)
- Convergence: $\|\varepsilon\| \le \left\|\hat{\mathcal{L}}_{\Delta x}^{-1}\right\| \, \|\tau_{\Delta x}\| \le \alpha \, O(\Delta x^p)$

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