Complete the Online Evaluation of 2.29

The spring 2015 end-of-term subject evaluations

opened on Tuesday, May 5 at 9 am.

Students have until Monday, May 18 at 9 am to complete their surveys.

Email TA to get two bonus points (surveys are anonymous)

2.29 Numerical Fluid Mechanics
Spring 2015 – **Lecture 24**

REVIEW Lecture 23:

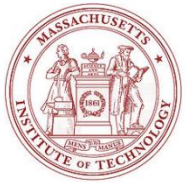
- Finite Element Methods
  - Introduction

$$\tilde{u}(x) = \sum_{i=1}^{n} a_i \, \phi_i(x) \quad \Rightarrow \quad L\left(\tilde{u}(x)\right) - f(x) = R(x) \neq 0$$

  - Method of Weighted Residuals: Galerkin, Subdomain and Collocation
  - General Approach to Finite Elements:

$$\int_t \int_V R(\mathbf{x}) \, w_i(\mathbf{x}) \, d\mathbf{x} \, dt = 0, \quad i = 1, 2, \ldots, n$$

    - Steps in setting-up and solving the discrete FE system
    - Galerkin Examples in 1D and 2D
  - Computational Galerkin Methods for PDE: general case
    - Variations of MWR: summary
    - Finite Elements and their basis functions on local coordinates (1D and 2D)
    - Isoparametric finite elements and basis functions on local coordinates (1D, 2D, triangular)
  - High-Order: Motivation
  - Continuous and Discontinuous Galerkin FE methods:
    - CG vs. DG
    - Hybridizable Discontinuous Galerkin (HDG): Main idea and example
  - DG: Worked simple example



TODAY (Lecture 24):
Finite Volume on Complex Geometries
Turbulent Flows and their Numerical Modeling

- Finite Volume on Complex geometries
  - Computation of convective fluxes
  - Computation of diffusive fluxes
  - Comments on 3D

- Turbulent Flows and their Numerical Modeling
  - Properties of Turbulent Flows
    - Stirring and Mixing
    - Energy Cascade and Scales
  - Turbulent Wavenumber Spectrum and Scales
  - Numerical Methods for Turbulent Flows: Classification
  - Direct Numerical Simulations (DNS) for Turbulent Flows
  - Reynolds-averaged Navier-Stokes (RANS)
  - Large-Eddy Simulations (LES)

References and Reading Assignments

- Chapter 8 on “Complex Geometries” of “J. H. Ferziger and M. Peric, Computational Methods for Fluid Dynamics. Springer, NY, 3rd edition, 2002”

- Chapter 9 on “Turbulent Flows” of “J. H. Ferziger and M. Peric, Computational Methods for Fluid Dynamics. Springer, NY, 3rd ed., 2002”
- Chapter 3 on “Turbulence and its Modelling” of H. Versteeg, W. Malalasekra, An Introduction to Computational Fluid Dynamics: The Finite Volume Method. Prentice Hall, Second Edition.
- Chapter 4 of “I. M. Cohen and P. K. Kundu. Fluid Mechanics. Academic Press, Fourth Edition, 2008”
- Chapter 3 on “Turbulence Models” of T. Cebeci, J. P. Shao, F. Kafyeke and E. Laurendeau, Computational Fluid Dynamics for Engineers. Springer, 2005

Finite Volumes on Complex geometries

- FD method (classic):
  - Use structured-grid transformation (either algebraic-transfinite, general, differential or conformal mapping)
  - Solve transformed equations on simple orthogonal computational domain
- FV method:
  - Starts from conservation eqns. in integral form on CV

$$\frac{d}{dt}\int_{CV}\rho\phi dV + \underbrace{\int_{CS}\rho\phi\,(\vec{v}.\vec{n})dA}_{\text{Advective (convective) fluxes}} = \underbrace{-\int_{CS}\vec{q}_{\phi}.\vec{n}\;dA}_{\text{Other transports (diffusion, etc)}} + \underbrace{\sum\int_{CV}s_{\phi}\;dV}_{\substack{\text{Sum of sources and}\\ \text{sinks terms (reactions, etc)}}}$$

  - We have seen principles of FV discretization
    - Convective/diffusive fluxes, from 1st - 2nd order to higher order discretizations
    - These principles are independent of grid specifics, but,
    - Several new features arise with non-orthogonal or arbitrary unstructured grids

Expressing fluxes at the surface based on cell-averaged (nodal) values: Summary of Two Approaches and Boundary Conditions

- Set-up of surface/volume integrals: 2 approaches (do things in opposite order)

1. (i) Evaluate integrals using classic rules (symbolic evaluation); (ii) Then, to obtain the unknown symbolic values, interpolate based on cell-averaged (nodal) values

$$\left.\begin{aligned} &(i)\ F_e = \int_{S_e} f_\phi\, dA \quad \Rightarrow \quad F_e = \mathcal{G}(\phi_e) \\ &(ii)\ \phi_e = \mathcal{H}(\bar{\phi}_P\text{'}s) \equiv \mathcal{H}(\phi_P\text{'}s) \end{aligned}\right\} \Rightarrow F_e = \mathcal{F}(\bar{\phi}_P\text{'}s)$$

Similar for other integrals:

$$(S_\phi = \int_V s_\phi\, dV\ ,\ \ \bar{\Phi} = \frac{1}{V}\int_V \rho\phi dV\ ,\ etc)$$

2. (i) Select shape of solution within CV (piecewise approximation); (ii) impose volume constraints to express coefficients in terms of nodal values; and (iii) then integrate. (this approach was used in the examples).

$$\left.\begin{aligned} &(i)\ \phi_{a_i}(x) \equiv \mathcal{J}_{a_i}(x) \\ &(ii)\ \int_{V_P} \phi_{a_i}(x) \equiv \bar{\phi}_P \end{aligned}\right\} \Rightarrow \phi_{a_i}(x) \equiv \phi_{\bar{\phi}_P}(x) \left.\begin{aligned} &\\ &\\ &(iii)\ F_e = \int_{S_e} f_{\phi_{\bar{\phi}P}}\, dA \end{aligned}\right\} \Rightarrow F_e = \mathcal{F}(\bar{\phi}_P\text{'}s)$$

Similar for higher dimensions:

$$\phi(x,y) \equiv \mathcal{J}_{a_i}(x,y);\quad etc$$

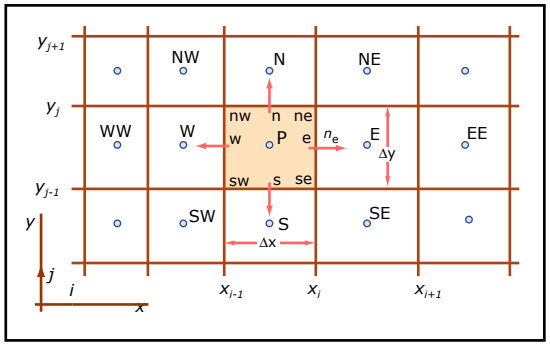
$$\phi_{a_i}(x_P, y_P) \equiv \phi_P;\quad etc$$

- Boundary conditions:
  - Directly imposed for convective fluxes
  - One-sided differences for diffusive fluxes

(From lecture 16)

Approx. of Surface/Volume Integrals: Classic symbolic formulas

(summary from Lecture 15)



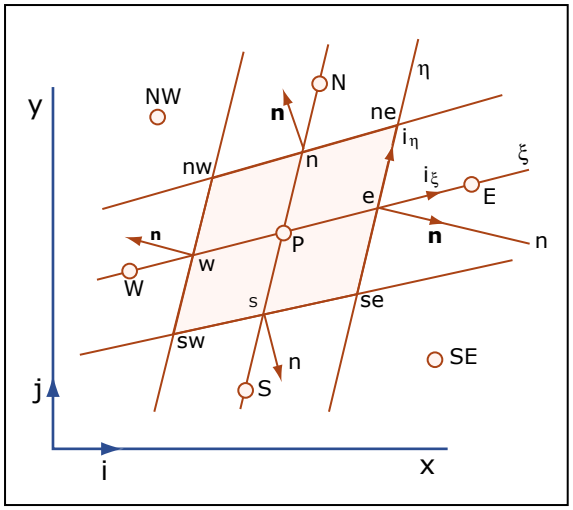
Notation used for a Cartesian 2D and 3D grid.
Image by MIT OpenCourseWare.

- Surface Integrals $F_e = \int_{S_e} f_\phi \, dA$
  - 2D problems (1D surface integrals)
    - Midpoint rule (2nd order): $F_e = \int_{S_e} f_\phi \, dA = \bar{f}_e S_e = f_e S_e + O(\Delta y^2) \approx f_e S_e$
    - Trapezoid rule (2nd order): $F_e = \int_{S_e} f_\phi \, dA \approx S_e \frac{(f_{ne} + f_{se})}{2} + O(\Delta y^2)$
    - Simpson's rule (4th order): $F_e = \int_{S_e} f_\phi \, dA \approx S_e \frac{(f_{ne} + 4 f_e + f_{se})}{6} + O(\Delta y^4)$
  - 3D problems (2D surface integrals)
    - Midpoint rule (2nd order): $F_e = \int_{S_e} f_\phi \, dA \approx S_e f_e \quad + O(\Delta y^2, \Delta z^2)$
    - Higher order more complicated to implement in 3D
- Volume Integrals: $S_\phi = \int_V s_\phi \, dV$, $\quad \bar{\Phi} = \frac{1}{V} \int_V \rho \phi dV$
  - 2D/3D problems, Midpoint rule (2nd order): $S_P = \int_V s_\phi \, dV = \bar{s}_P \, V \approx s_P \, V$
  - 2D, bi-quadratic (4th order, Cartesian): $S_P = \frac{\Delta x \, \Delta y}{36} [16 s_P + 4 s_s + 4 s_n + 4 s_w + 4 s_e + s_{se} + s_{sw} + s_{ne} + s_{nw}]$

FV: Approximation of **convective fluxes**

$$\underbrace{\int_{CS} \rho\phi\,(\vec{v}.\vec{n})\,dS}_{\text{Advective (convective) fluxes}}$$

- For complex geometries, one often uses the midpoint rule for the approximation of surface and volume integrals
- Consider first the mass flux: $\phi = 1$: $f_{\phi=1} = \rho\,\vec{v}.\vec{n}$
  - Again, we consider one face only: east side of a 2D CV (same approach applies to other faces and to any CV shapes).
  - Mid-point rule for mass flux: $\dot{m}_e = \int_{S_e} f_{\phi=1}\,dS = \bar{f}_e S_e = f_e S_e + O(\Delta^2) \approx (\rho\,\vec{v}.\vec{n})_e S_e$



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- The unit normal vector to face "e" and its surface S_e are defined as: $\mathbf{n}_e S_e = S_e^x\,\mathbf{i} + S_e^y\,\mathbf{j} = (y_{ne} - y_{se})\,\mathbf{i} - (x_{ne} - x_{se})\,\mathbf{j}$

  where $S_e = \sqrt{(S_e^x)^2 + (S_e^y)^2}$

- Hence, mass flux is:

$$\dot{m}_e \approx (\rho\,\vec{v}.\vec{n})_e S_e = \rho_e\,\vec{v}_e.(S_e^x\,\mathbf{i} + S_e^y\,\mathbf{j}) = \rho_e\,(S_e^x\,u_e + S_e^y\,v_e)$$

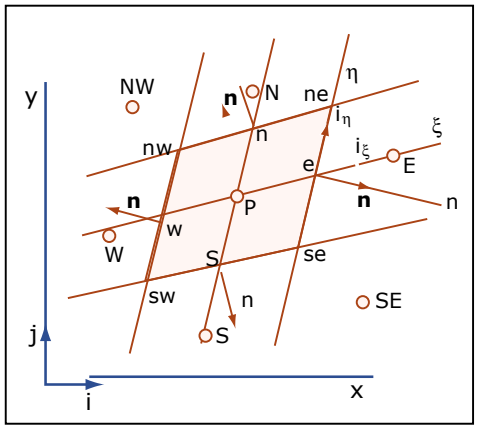
FV: Approximation of convective fluxes, Cont'd
Mass Flux

- The mass flux for the mid-point rule: $\dot{m}_e = \rho_e \, (S_e^x \, u_e + S_e^y \, v_e)$

- What's the difference between the Cartesian and non-orthogonal grid cases?

  – In non-orthogonal case, normal to surface has components in all directions

  – All velocity components thus contribute to the flux (each component is multiplied by the projection of S_e onto the corresponding axis)



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FV: Approximation of convective fluxes, Cont'd

- Mass flux for mid-point rule: $\dot{m}_e = \rho_e \, (S_e^x \, u_e + S_e^y \, v_e)$
- Convective flux for any transported ϕ
  - Is usually computed after the mass flux. Using the mid-point rule:

$$F_e = \int_{S_e} \rho\phi \, (\vec{v}.\vec{n}) \, dS \approx f_e S_e = (\rho\phi\vec{v}.\vec{n})_e S_e = \phi_e \dot{m}_e$$

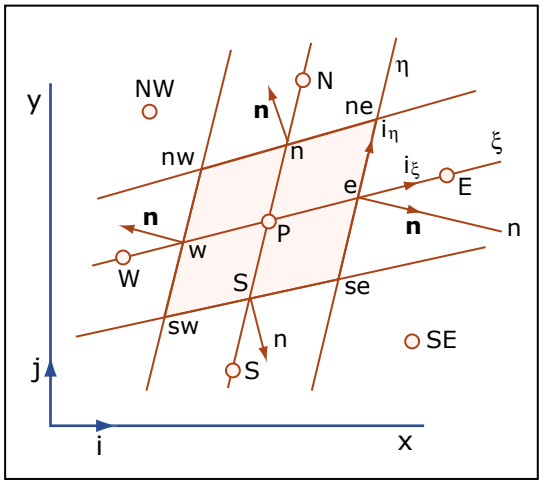
where ϕ_e = value ϕ at center of cell face

  - How to obtain ϕ_e?, use either:
    - A linear interpolation between two nodes on either side of face (also 2nd order) $\Rightarrow$ becomes trapezoidal rule
    - Fit ϕ to a polynomial in the vicinity of the face (piecewise shape function)
  - Considerations for unstructured grid:
    - Best compromise among accuracy, generality and simplicity is usually: Linear interpolation and mid-point rule
    - Indeed: facilitates use of local grid refinement, which can be used to achieve higher accuracy at lower cost than higher-order schemes. However, higher-order FE or compact FD are now being used/developed

FV: Approximation of **diffusive fluxes**

$$\underbrace{\int_{CS} k\nabla\phi.\vec{n}\, dS}_{\text{Diffusive Fluxes}}$$

- For complex geometries, we can still use the midpoint rule
- Mid-point rule gives: $F_e^d = \int_{S_e} k\nabla\phi.\vec{n}\, dS = \overline{f}_e S_e = f_e S_e + O(\Delta^2) \approx (k\nabla\phi.\vec{n})_e S_e$
- Here, gradient can be expressed in terms of global Cartesian coordinates (x, y) or local orthogonal coordinates (n, t)



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– In 2D: $\nabla\phi = \frac{\partial\phi}{\partial x}\mathbf{i} + \frac{\partial\phi}{\partial y}\mathbf{j} = \frac{\partial\phi}{\partial n}\mathbf{n} + \frac{\partial\phi}{\partial t}\mathbf{t}$

– There are many ways to approximate the derivative normal to the cell face or the gradient vector at the cell center

– As always, two main approaches:

- Approximate surface integral, then interpolate
- Specify shape function, constraints, then integrate

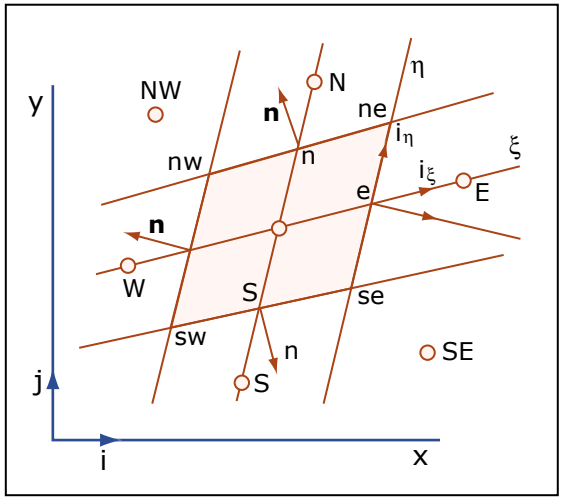
FV: Approximation of **diffusive fluxes, Cont'd**

1) If shape function $\phi(x, y)$ is used, with mid-point rule, this gives:

$$F_e^d \approx (k\nabla\phi.\vec{n})_e S_e = k_e \left(S_e^x \left.\frac{\partial\phi}{\partial x}\right|_e + S_e^y \left.\frac{\partial\phi}{\partial y}\right|_e \right) = k_e \sum_i S_e^{x_i} \left.\frac{\partial\phi}{\partial x_i}\right|_e$$

- Can be evaluated and relatively easy to implement explicitly
- Implicitly can be harder for high-order shape fct $\phi(x, y)$ (more cell involved)

2) Another way is to compute derivatives at CV centers first, then interpolate to cell faces (2 steps as for computing ϕ_e from ϕ_P)



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i) One can use averages + Gauss Theorem locally

- Derivative at center ≈ average derivative over cell

$$\left.\frac{\partial\phi}{\partial x}\right|_P \approx \int_{CV} \frac{\partial\phi}{\partial x} dV \Big/ dV = \left.\overline{\frac{\partial\phi}{\partial x}}\right|_P$$

- Gauss theorem for $\partial\phi/\partial x$ (similar for y derivative):

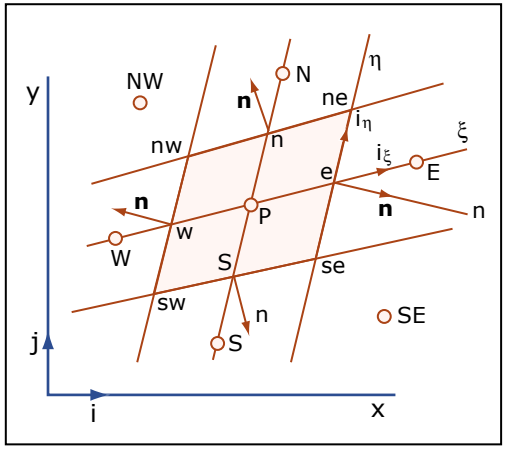
$$\int_{CV} \frac{\partial\phi}{\partial x} dV = \int_{CS} \phi\, \mathbf{i}.\mathbf{n}\, dS \approx \sum_{4\text{faces } c} \phi_c S_c^x$$

FV: Approximation of **diffusive fluxes, Cont'd**

- Hence, the gradient at the CV center with respect to x_i is obtained by summing the products of each ϕ_c with the projection of its cell surface onto a plane normal to x_i, and then dividing the sum by the CV volume

$$\left.\frac{\partial \phi}{\partial x_i}\right|_P \approx \left.\overline{\frac{\partial \phi}{\partial x_i}}\right|_P = \int_{CV} \frac{\partial \phi}{\partial x_i} dV \Big/ dV \approx \sum_{4\,c\ \text{faces}} \phi_c S_c^{x_i} \Big/ dV$$

- For ϕ_c we can use the approximation for the convective fluxes
- We can then interpolate to obtain the gradient at the centers of cell faces
- For Cartesian grids and linear interpolation, one retrieves centered FD



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ii) Cell-center gradients can also be approximated to 2nd order assuming a linear variation of ϕ locally:

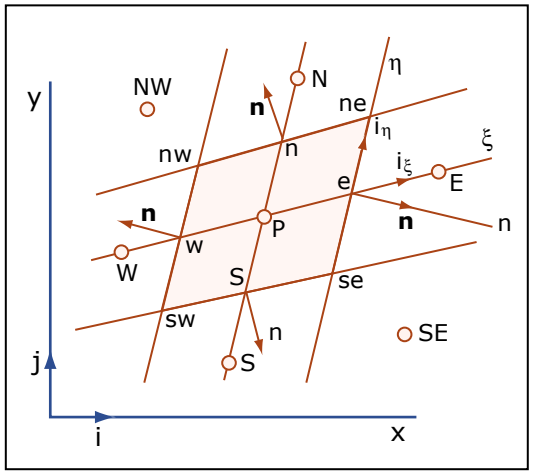
$$\phi_E - \phi_P \approx \nabla\phi\big|_P . (\mathbf{r}_E - \mathbf{r}_P)$$

- There are as many such equations as there are neighbors to the cell centered at $P \Rightarrow$ need for least-square inversions (only *n* derivatives in *n*D)
- Issues with this approximation are oscillatory solutions and large computational stencils for implicit schemes $\Rightarrow$ use deferred-correction approach

FV: Approximation of **diffusive fluxes, Cont'd**

iii) Deferred-Correction Approach:

- Idea behind deferred-correction is to identify possible options and combine them to reduce costs and eliminate un-desired behavior. Some options:
- If we work in local coordinates (n, t): $F_e^d \approx (k\nabla\phi.\vec{n})_e S_e = k_e S_e \left.\frac{\partial\phi}{\partial n}\right|_e$
- If grid was close to orthogonal Cartesian, using CDS: $\left.\frac{\partial\phi}{\partial n}\right|_e \approx \left.\frac{\partial\phi}{\partial\xi}\right|_e \approx \frac{\phi_E - \phi_P}{|\mathbf{r}_E - \mathbf{r}_P|}$ (1)
- If interpolate the gradient at the cell center: $\left.\frac{\partial\phi}{\partial n}\right|_e \approx \left.\frac{\partial\phi}{\partial\xi}\right|_e \approx \frac{1}{2}\frac{\phi_E - \phi_W}{|\mathbf{r}_E - \mathbf{r}_W|} + \frac{1}{2}\frac{\phi_{EE} - \phi_P}{|\mathbf{r}_{EE} - \mathbf{r}_P|}$ (2)



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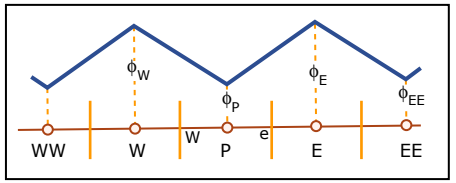
- A fast oscillatory solution doesn't contribute to this 3rd higher-order choice, but gradients at cell faces would then be large => oscillations do occur:



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- The obvious solution: $\left.\frac{\partial\phi}{\partial n}\right|_e \approx \left[(1)^{\text{implicit}}\right]^{r+1} + \left[(2)^{\text{explicit}} - (1)^{\text{implicit}}\right]^{r}$ will oscillate $\Rightarrow$ Need to find other options/solution

FV: Approximation of **diffusive fluxes, Cont'd**

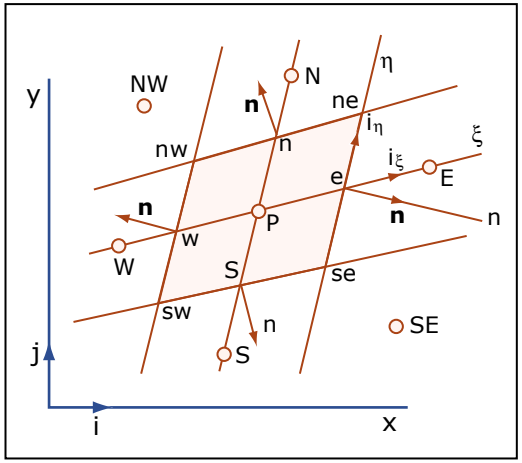
iii) Deferred-Correction Approach, Cont'd (Muzaferija, 1994):

– If line connecting nodes P and E is nearly orthogonal to cell face, derivative w.r.t n can be approximated with derivative w.r.t ξ (as before):

- $\Rightarrow$ approximation close to 2nd order $F_e^d \approx k_e S_e \left.\frac{\partial\phi}{\partial n}\right|_e \approx k_e S_e \left.\frac{\partial\phi}{\partial\xi}\right|_e \approx k_e S_e \frac{\phi_E - \phi_P}{|\mathbf{r}_E - \mathbf{r}_P|}$

– If grid is not orthogonal, the deferred correction term should contain the difference between the gradients in the n and ξ directions $\Rightarrow$

$$F_e^d \approx k_e S_e \left.\frac{\partial\phi}{\partial\xi}\right|_e + k_e S_e \left[\left.\frac{\partial\phi}{\partial n}\right|_e - \left.\frac{\partial\phi}{\partial\xi}\right|_e\right]^{\text{old}}$$



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- where the first term is computed as: $k_e S_e \left.\frac{\partial\phi}{\partial\xi}\right|_e = k_e S_e \frac{\phi_E - \phi_P}{|\mathbf{r}_E - \mathbf{r}_P|}$
- bracket term is interpolated from cell center gradients (themselves obtained from Gauss theorem)

$$\left.\frac{\partial\phi}{\partial n}\right|_e \approx \left.\overline{\frac{\partial\phi}{\partial n}}\right|_e = \left.\overline{\nabla\phi}\right|_e . \mathbf{n} \quad \text{and} \quad \left.\frac{\partial\phi}{\partial\xi}\right|_e \approx \left.\overline{\frac{\partial\phi}{\partial\xi}}\right|_e = \left.\overline{\nabla\phi}\right|_e . \mathbf{i}_\xi$$

- Hence:

$$F_e^d \approx k_e S_e \frac{\phi_E - \phi_P}{|\mathbf{r}_E - \mathbf{r}_P|} + k_e S_e \left[\left.\overline{\nabla\phi}\right|_e\right]^{\text{old}} \left(\mathbf{n} - \mathbf{i}_\xi\right)$$

FV: Approximation of **diffusive fluxes, Cont'd**

iii) Deferred-Correction Approach, Cont'd (Muzaferija, 1994):

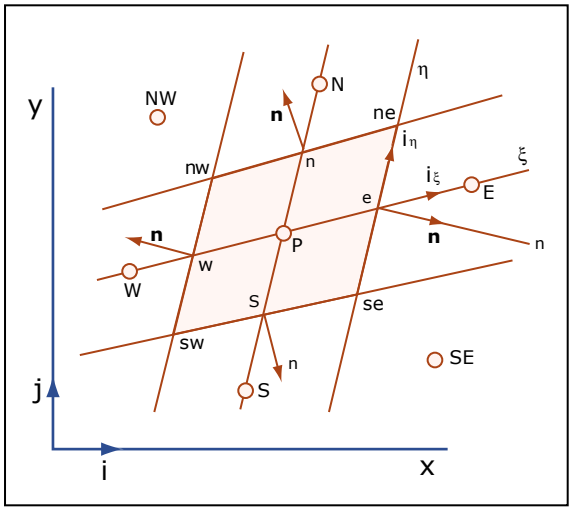
– In the formula:

$$F_e^d \approx k_e S_e \frac{\phi_E - \phi_P}{|\mathbf{r}_E - \mathbf{r}_P|} + k_e S_e \left[\overline{\nabla\phi}\big|_e\right]^{\text{old}} \left(\mathbf{n} - \mathbf{i}_\xi\right)$$

- The deferred correction term is (close to) zero when grid (close to) orthogonal, i.e. n and ξ directions are the same (close to each other).
- It makes the computation of derivatives simple (amounts to sums of neighbor values), recall that:

$$\left[\overline{\nabla\phi}\big|_e\right]^{\text{old}} \text{ interpolated from } \left[\overline{\nabla\phi}\big|_P\right]^{\text{old}},$$

$$\text{the latter given by e.g. } \overline{\frac{\partial\phi}{\partial x_i}}\bigg|_P = \sum_{4\,c\ \text{faces}} \phi_c S_c^{x_i} \Big/ dV$$



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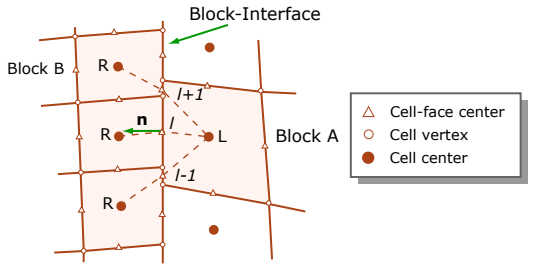
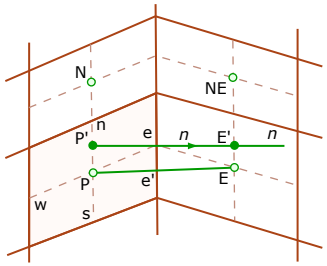
- Prevents oscillations since based on sums of ϕ_c, with positive coefficients
- We remained in Cartesian coordinates (no need to transform coordinates, we just need to know the normals & surfaces), which is handy for complex turbulent models

Some comments on FV on complex geometries

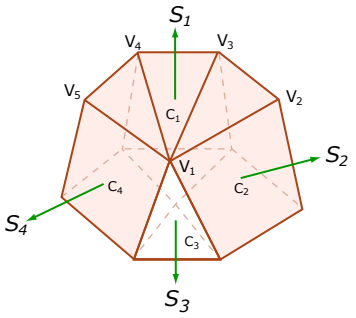
- Line P-E does not always pass through the cell center
  - $\Rightarrow$ need some updates in that case
  - otherwise, scheme is of lower order (e.g. approximation is not second order anymore)
- Schemes can be extended to 3D grids but some updates can also be needed
  - For example, cell faces are not always planar, harder in 3D
- Block-structured grids and nested grids also need special treatment
  - For example, matching at boundaries (usually interpolation and averaging)



The interface between two blocks with non-matching grids.



Collocated arrangement of velocity components and pressure on FD and FV grids.



Cell volume and surface vectors for arbitrary control volumes.

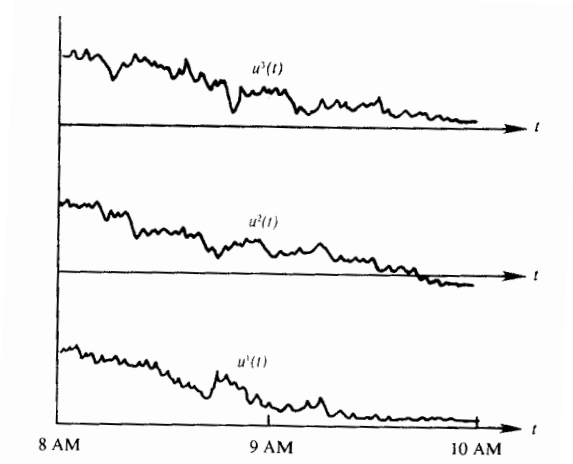
Image by MIT OpenCourseWare.

Turbulent Flows and their Numerical Modeling

- Most real flows are turbulent (at some time and space scales)
- Properties of turbulent flows
  - Highly unsteady: velocity at a point appears random
  - Three-dimensional in space: instantaneous field fluctuates rapidly, in all three dimensions (even if time-averaged or space-averaged field is 2D)

Some Definitions

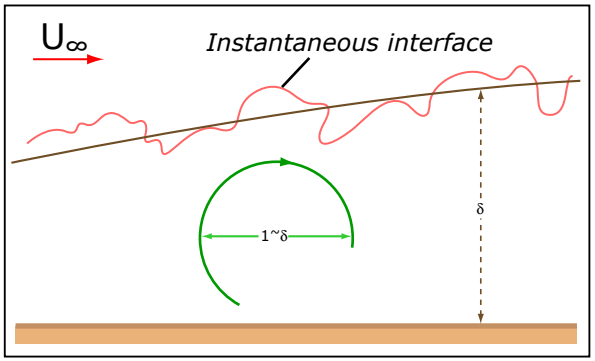
- Ensemble averages: "average of a collection of experiments performed under identical conditions"
- Stationary process: "statistics independent of time"
- For a stationary process, time and ensemble averages are equal



Three turbulent velocity realizations in an atmospheric BL in the morning (Kundu and Cohen, 2008)

Turbulent Flows and their Numerical Modeling

- Properties of turbulent flows, Cont'd
  - Highly nonlinear (e.g. high Re)
  - High vorticity: vortex stretching is one of the main mechanisms to maintain or increase the intensity of turbulence
  - High stirring: turbulence increases rate at which conserved quantities are stirred
    - Stirring: advection process by which conserved quantities of different values are brought in contact (swirl, folding, etc)
    - Mixing: irreversible molecular diffusion (dissipative process). Mixing increases if stirring is large (because stirring leads to large 2nd and higher spatial derivatives).
    - Turbulent diffusion: averaged effects of stirring modeled as "diffusion"
  - Characterized by "Coherent Structures"
    - CS are often spinning, i.e. eddies
    - Turbulence: wide range of eddies' size, in general, wide range of scales



Turbulent flow in a BL: Large eddy has the size of the BL thickness

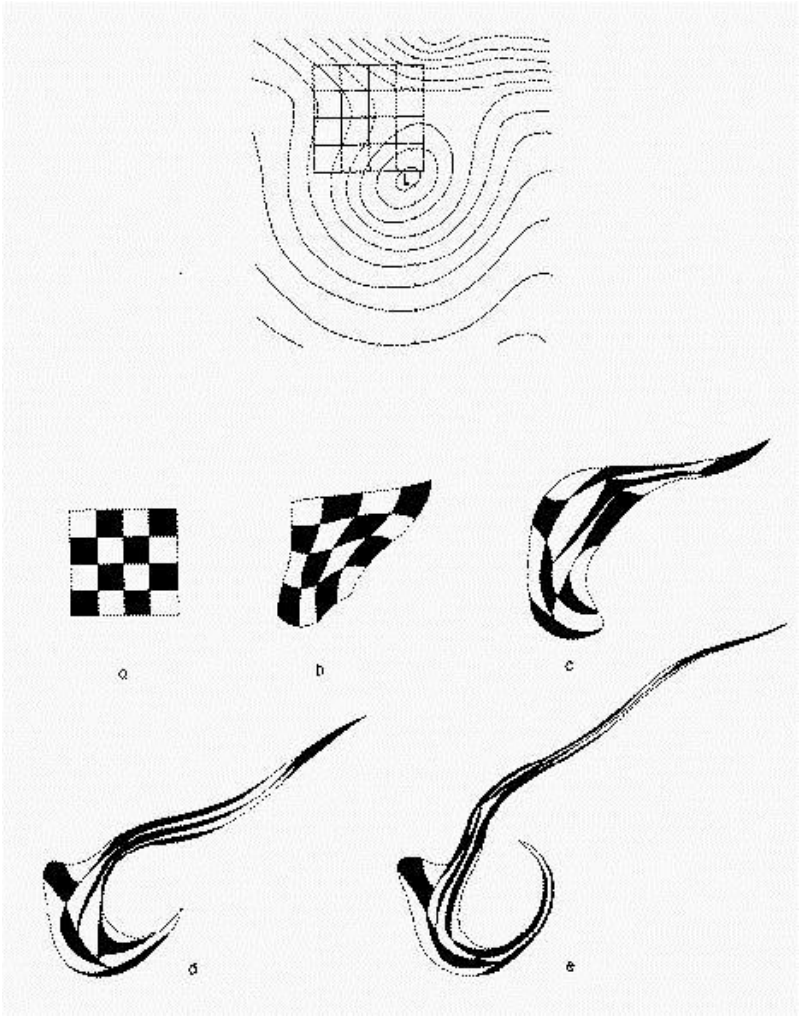
Image by MIT OpenCourseWare.

Stirring and Mixing

Welander's "scrapbook"

Welander P. Studies on the general development of motion in a two-dimensional ideal fluid. *Tellus, 7:141–156, 1955.*

- His numerical solution illustrates differential advection by a simple velocity field.
- A checkerboard pattern is deformed by a numerical quasigeostrophic barotropic flow which models atmospheric flow at the 500mb level. The initial streamline pattern is shown at the top. Shown below are deformed check board patterns at 6, 12, 24 and 36 hours, respectively.
- Notice that each square of the checkerboard maintains constant area as it deforms (conservation of volume).



Energy Cascade and Scales

British meteorologist Richarson's famous quote:

"Big whorls have little whorls,
Which feed on their velocity,
And little whorls have lesser whorls,
And so on to viscosity".

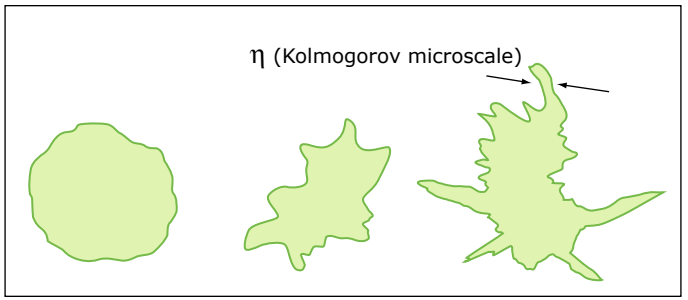


Image by MIT OpenCourseWare.

Dimensional Analyses and Scales (Tennekes and Lumley, 1972, 1976)

- Largest eddy scales L, T, U: L/T= Distance/time over which fluctuations are correlated and U = large eddy velocity (usually all three are close to scales of mean flow)
- Viscous scales: η, τ, u_v = viscous length (Kolmogorov scale), time and velocity scales

Hypothesis: rate of turbulent energy production ≈ rate of viscous dissipation

- Length-scale ratio: $L/\eta \sim O(\mathrm{Re}_L^{3/4}) \qquad \mathrm{Re}_L = UL/\nu$
- Time-scale ratio: $T/\tau \sim O(\mathrm{Re}_L^{1/2})$
- Velocity-scale ratio: $U/u_v \sim O(\mathrm{Re}_L^{1/4})$

Re_L= largest eddy Re
~ Re_{mean} to 0.01 Re_{mean}

Turbulent Wavenumber Spectrum and Scales

- Turbulent Kinetic Energy Spectrum $S(K)$: $\overline{u'^2} = \int_0^\infty S(K)\,dK$
- In the inertial sub-range, Kolmogorov argued by dimensional analysis that

$$S = S(K,\varepsilon) = A\,\varepsilon^{2/3}K^{-5/3} \qquad \ell^{-1} \ll K \ll \eta^{-1}$$

$A \simeq 1.5$ found to be universal for turbulent flows

- Turbulent energy dissipation

$$\varepsilon \sim \frac{\text{Turb. energy}}{\text{Turb. time scale}} = u'^2 \times \frac{u'}{L} = \frac{u'^3}{L}$$

- Komolgorov microscale:
  - Size of eddies depend on turb. dissipation ε and viscosity ν
  - Dimensional Analysis: $\eta \sim \left(\frac{\nu^3}{\varepsilon}\right)^{1/4}$

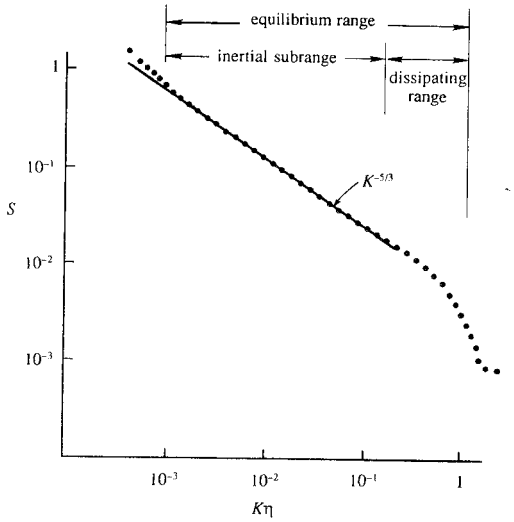


Figure 13.12 A typical wavenumber spectrum observed in the ocean, plotted on a log–log scale. The unit of S is arbitrary, and the dots represent hypothetical data.

$\ell = L$ (large eddies scale)

$u = U + u'$

Numerical Methods for Turbulent Flows

Primary approach (used to be) is experimental

Numerical Methods classified into methods based on:

1) Correlations: useful mostly for 1D problems, e.g.: $f = f(\mathrm{Re}, \varepsilon)$ $Nu = \varphi(\mathrm{Re}, \mathrm{Pr}, Ra)$

- Moody chart or friction factor relations for turbulent pipe flows, Nusselt number for heat transfer as a function of Re and Pr, etc.

2) Integral equations:

- Integrate PDEs (NS eqns.) in one or more spatial coordinates
- Solve using ODE schemes (time-marching)

3) Averaged equations

- Averaged over time or over an (hypothetical) ensemble of realizations
- Often decompositions into mean and fluctuations: $u = \bar{u} + u'$; $\phi = \bar{\phi} + \phi'$
- Require closure models and lead to a set of PDEs: Reynolds-averaged Navier-Stokes (RANS) eqns. "*One-(spatial) point closure*" methods

Numerical Methods for Turbulent Flows

Numerical Methods classification, Cont'd:

4) Large-Eddy Simulations (LES)

- Solves for the largest scales of motions of the flow
- Only approximates or parameterizes the small scale motions
- Compromise between RANS and DNS

5) Direct Numerical Simulations (DNS)

- Solves for all scales of motions of the turbulent flow (full Navier-Stokes)

- The methods 1) to 5) make less and less approximations, but computational time increases from 1) to 5).
- Conservation PDEs are solved as for laminar flows: major challenge is the much wider range of scales (of motions, heat transfer, etc)

2.29 Numerical Fluid Mechanics

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