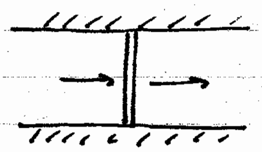
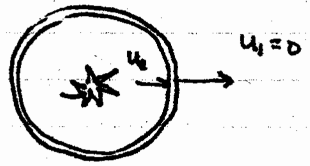
Normal shocks





All shocks can locally be changed into normal shock by transforming to a moving frame.

Recall Rankine-Hugoniot for a perfect gas

$$\frac{P_2}{P_1} = \frac{\frac{\gamma+1}{\gamma-1} - \frac{v_2}{v_1}}{\frac{\gamma+1}{\gamma-1}\frac{v_2}{v_1} - 1}$$

Rearrange: $\left[\frac{\gamma+1}{\gamma-1} v_2 - v_1\right] P_2 = \left[\frac{\gamma+1}{\gamma-1} v_1 - v_2\right] P_1$

$$\left(\frac{\gamma+1}{\gamma-1} P_2 + P_1\right)[v] = \left(\frac{\gamma+1}{\gamma-1} P_1 + P_2 - \frac{\gamma+1}{\gamma-1} P_2 - P_1\right) v_1$$

$$\frac{[v]}{v_1} = -\frac{2[P]/P_1}{2\gamma + (\gamma+1)[P]/P_1} \qquad \text{(perfect gas)}$$

As usual, we are looking for expressions for [ ] as a fn of Mach # normalized by the appropriate state (state ①). (Recall we have written state variables as ratios to stagnation properties; now we would like ratios in the form [ ]/①)

From last time: $\frac{[P]}{\rho_1 c_1^2} = -M_{in}\frac{[w]}{c_1} = -M_{in}^2 \frac{[v]}{v_1}$

Combining w. $\frac{[v]}{v_1}$ for a perfect gas we get all of the desired relations:

$$\boxed{\begin{aligned} \frac{[P]}{P_1} &= \frac{2\gamma}{\gamma+1}\left(M_{1n}^2 - 1\right) \\ \frac{[w]}{c_1} &= -\frac{2}{\gamma+1}\left(M_{1n} - \frac{1}{M_{1n}}\right) \\ \frac{[v]}{v_1} &= -\frac{2}{\gamma+1}\left(1 - \frac{1}{M_{1n}^2}\right) \end{aligned}}$$

given state 1; these fix thermodynamic state + velocities downstream.

Tabulated in D.2 for $\gamma = 1.4$ (show Fig. 7.14)

Rearranging: $M_{2n} = w_2/c_1 = \frac{w_1 + [w]}{c_1} \frac{c_1}{c_2}$

After algebra (see pg. 324) $\boxed{M_{2n}^2 = \frac{(\gamma-1) M_{1n}^2 + 2}{2\gamma M_{1n}^2 - (\gamma-1)}}$

Strong shock: (weak shocks last time...)

$$[P] \gg 1 \Rightarrow M_{1n}^2 \gg 1$$

$$\text{As } M_{1n}^2 \to \infty \Rightarrow M_{2n}^2 \to \left(\frac{\gamma-1}{2\gamma}\right)$$

The following are equivalent criteria for a strong shock:

$$\left.\begin{aligned} &\Pi = \frac{[P]}{\rho_1 c_1^2} \\ &-M_{1n} \frac{[w]}{c_1} \\ &M_{1n}^2 \end{aligned}\right\} \gg 1$$

Expect P_2 to dominate:

$$\frac{[P]}{\rho_1 c_1^{\cancel{2}}} = -M_{1n} \frac{[w]}{\cancel{c_1}} \Rightarrow P_2 \sim -M_{1n} \rho_1 c_1 [w]$$

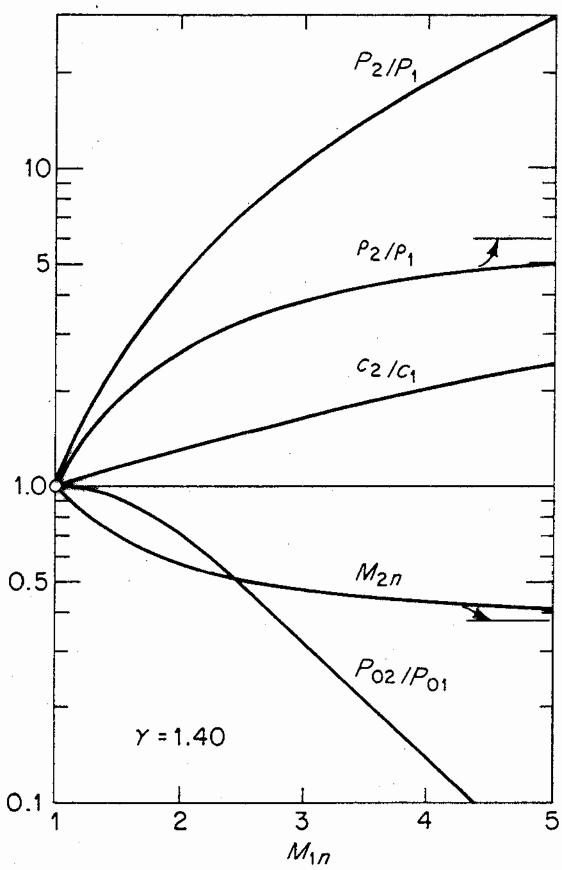


Figure 7.14
Downstream conditions as a function of the shock Mach number for a perfect gas with $\gamma = 1.40$.

$$P_2 \sim -\underset{\rho w \text{ by cons of mass}}{\underbrace{\rho_1 w_1}} [w] \quad \Rightarrow \quad \boxed{P_2 = -\rho w [w]}$$

momentum transfer!

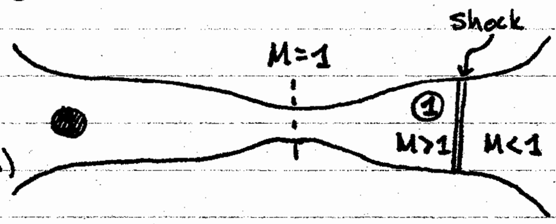
(downstream pressure from)

Limit $M_{1n}^2 \gg 1$ $(\to \infty)$ shock relations reduce to

$$\boxed{\begin{aligned} [w] &= -\frac{2}{\gamma+1} w_1 \\ P_2 &= \frac{2}{\gamma+1} \rho_1 w_1^2 \\ [v] &= \frac{-2}{\gamma+1} v_1 \\ M_{2n} &= \left(\frac{\gamma-1}{2\gamma}\right)^{1/2} \end{aligned}}$$

(Note: incident + reflected shock example in John's notes)

Entropy through shocks



$u_2 \approx 0$ (stagnation) but stagnation pressure is different from ①

Recall that the stagnation enthalpy is invariant across a stationary shock (from $[h + \tfrac{1}{2}w^2] = 0$)

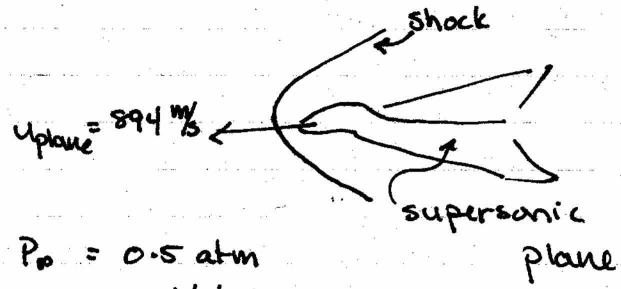
$\Rightarrow$ $T_{01} = T_{02}$ for a perfect gas.

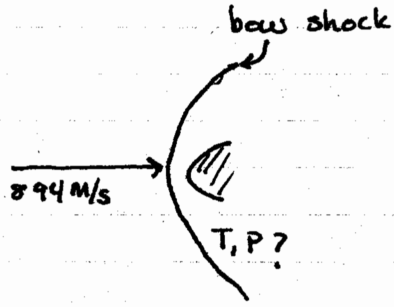
$$\Delta s = c_p \ln \cancel{\frac{T_2}{T_1}}^{\to 0} - R \ln \frac{P_2}{P_1}$$

$$\Rightarrow \frac{P_{02}}{P_{01}} = e^{-[s]/R} \quad \Rightarrow$$

- $P_{02} < P_{01}$
- can measure entropy change by measuring pressure ratio

Example (using normal shock relations)





$P_\infty = 0.5$ atm
$T_\infty = 244$ K

$$c_\infty = \sqrt{\gamma R T} \qquad R_{air} = 287.03 \ \frac{m^2}{s^2 K}$$

$$= \sqrt{(1.4)(287.03)(244)} = 313 \ m/s$$

$$\Rightarrow M_{in} = \frac{894}{313} = \boxed{2.85}$$

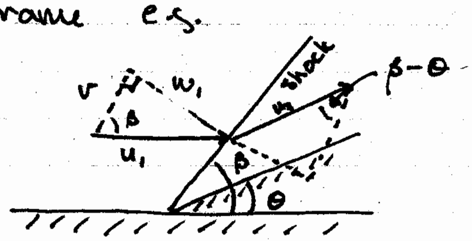
From shock tables (D.2) : $\frac{P_2}{P_\infty} = 9.310 \qquad \frac{T_2}{T_\infty} = 2.507$

$$\Rightarrow \boxed{P_2 = 4.65 \text{ atm} \qquad T_2 = 611 \text{ K}}$$

(Generally design aircraft to avoid normal shocks)

Oblique Shocks

Relative fluid velocities are "oblique" to the shock front. We can always change refence frams locally to make the shock normal. However, sometimes it is more natural to view the shock in an oblique ref. frame e.g.



(can we find $\beta(\theta)$?)

$\left\lvert\frac{[w]}{c_1}\right\rvert$	M_{2n}
1.455	0.5471
1.465	0.5457
1.475	0.5444
1.485	0.5431
1.495	0.5418
1.505	0.5406
1.515	0.5393
1.525	0.5381
1.535	0.5368
1.544	0.5356
1.554	0.5344
1.564	0.5332
1.574	0.5321
1.584	0.5309
1.594	0.5297
1.604	0.5286
1.614	0.5275
1.623	0.5264
1.633	0.5253
1.643	0.5242
1.653	0.5231
1.663	0.5221
1.672	0.5210
1.682	0.5200
1.692	0.5189
1.702	0.5179
1.711	0.5169
1.721	0.5159
1.731	0.5149
1.740	0.5140
1.750	0.5130
1.760	0.5120
1.769	0.5111
1.779	0.5102
1.789	0.5092
1.798	0.5083
1.808	0.5074
1.817	0.5065
1.827	0.5056
1.837	0.5047

M_{1n}	P_2/P_1	ρ_2/ρ_1	T_2/T_1	c_2/c_1	P_{02}/P_{01}	$\left\lvert\frac{[w]}{c_1}\right\rvert$	M_{2n}
2.60	7.720	3.449	2.238	1.496	0.460	1.846	0.5039
2.61	7.781	3.460	2.249	1.500	0.456	1.856	0.5030
2.62	7.842	3.471	2.259	1.503	0 453	1.865	0.5022
2.63	7.903	3.483	2.269	1.506	0.449	1.875	0.5013
2.64	7.965	3.494	2.280	1.510	0.445	1.884	0.5005
2.65	8.026	3.505	2.290	1.513	0.442	1.894	0.4996
2.66	8.088	3.516	2.301	1.517	0.438	1.903	0.4988
2.67	8.150	3.527	2.311	1.520	0.434	1.913	0.4980
2.68	8.213	3.537	2.322	1.524	0.431	1.922	0.4972
2.69	8.275	3.548	2.332	1.527	0.427	1.932	0.4964
2.70	8.338	3.559	2.343	1.531	0.424	1.941	0.4956
2.71	8.401	3.570	2.354	1.534	0.420	1.951	0.4949
2.72	8.465	3.580	2.364	1.538	0.417	1.960	0.4941
2.73	8.528	3.591	2.375	1.541	0.413	1.970	0.4933
2.74	8.592	3.601	2.386	1.545	0.410	1.979	0.4926
2.75	8.656	3.612	2.397	1.548	0.406	1.989	0.4918
2.76	8.721	3.622	2.407	1.552	0.403	1.998	0.4911
2.77	8.785	3.633	2.418	1.555	0.399	2.007	0.4903
2.78	8.850	3.643	2.429	1.559	0.396	2.017	0.4896
2.79	8.915	3.653	2.440	1.562	0.393	2.026	0.4889
2.80	8.980	3.664	2.451	1.566	0.389	2.036	0.4882
2.81	9.045	3.674	2.462	1.569	0.386	2.045	0.4875
2.82	9.111	3.684	2.473	1.573	0.383	2.054	0.4868
2.83	9.177	3.694	2.484	1.576	0.380	2.064	0.4861
2.84	9.243	3.704	2.496	1.580	0.376	2.073	0.4854
2.85	9.310	3.714	2.507	1.583	0.373	2.083	0.4847
2.86	9.376	3.724	2.518	1.587	0.370	2.092	0.4840
2.87	9.443	3.734	2.529	1.590	0.367	2.101	0.4833
2.88	9.510	3.743	2.540	1.594	0.364	2.111	0.4827
2.89	9.577	3.753	2.552	1.597	0.361	2.120	0.4820
2.90	9.645	3.763	2.563	1.601	0.358	2.129	0.4814
2.91	9.713	3.773	2.575	1.605	0.355	2.139	0.4807
2.92	9.781	3.782	2.586	1.608	0.352	2.148	0.4801
2.93	9.849	3.892	2.598	1.612	0.349	2.157	0.4795
2.94	9.918	3.801	2.609	1.615	0.346	2.167	0.4788
2.95	9.986	3.811	2.621	1.619	0.343	2.176	0.4782
2.96	10.055	3.820	2.632	1.622	0.340	2.185	0.4776
2.97	10.124	3.829	2.644	1.626	0.337	2.194	0.4770
2.98	10.194	3.839	2.656	1.630	0.334	2.204	0.4764
2.99	10.263	3.848	2.667	1.633	0.331	2.213	0.4758

$w_1 = u_1 \sin\beta$, $\quad v_1 = u_1 \cos\beta = v_2$

Supersonic upstream flow $\Rightarrow$ $M_1 \sin\beta \geq 1$ $\quad (u_1/c_1 = M_1)$

$$\therefore \boxed{M_1 \geq 1} \quad (M_{1n} = M_1 \sin\beta)$$

Downstream, the normal flow is subsonic (but u_2 may ~~##~~ not be. $\therefore$ $M_2 \gtrless 1$ (either is possible)

$$\tan\beta = \frac{w_1}{v} \qquad \tan(\beta - \theta) = \frac{w_2}{v}$$

Using these + trig + $\frac{[p]}{\rho_1 c_1^2} = \ldots$ and rearranging (pg. 328)

$$\frac{\Pi}{M_1^2} = \frac{\tan\theta}{\cot\beta + \tan\theta}$$

Combining this w. $\frac{[w]}{c_1} = \frac{-2}{\gamma+1}\left(M_{1n} - \frac{1}{M_{1n}}\right)$

$$\Rightarrow \boxed{\tan\theta = \frac{2\cot\beta\,(M_1^2 \sin^2\beta - 1)}{(\gamma+1)M_1^2 - 2(M_1^2\sin^2\beta - 1)}}$$

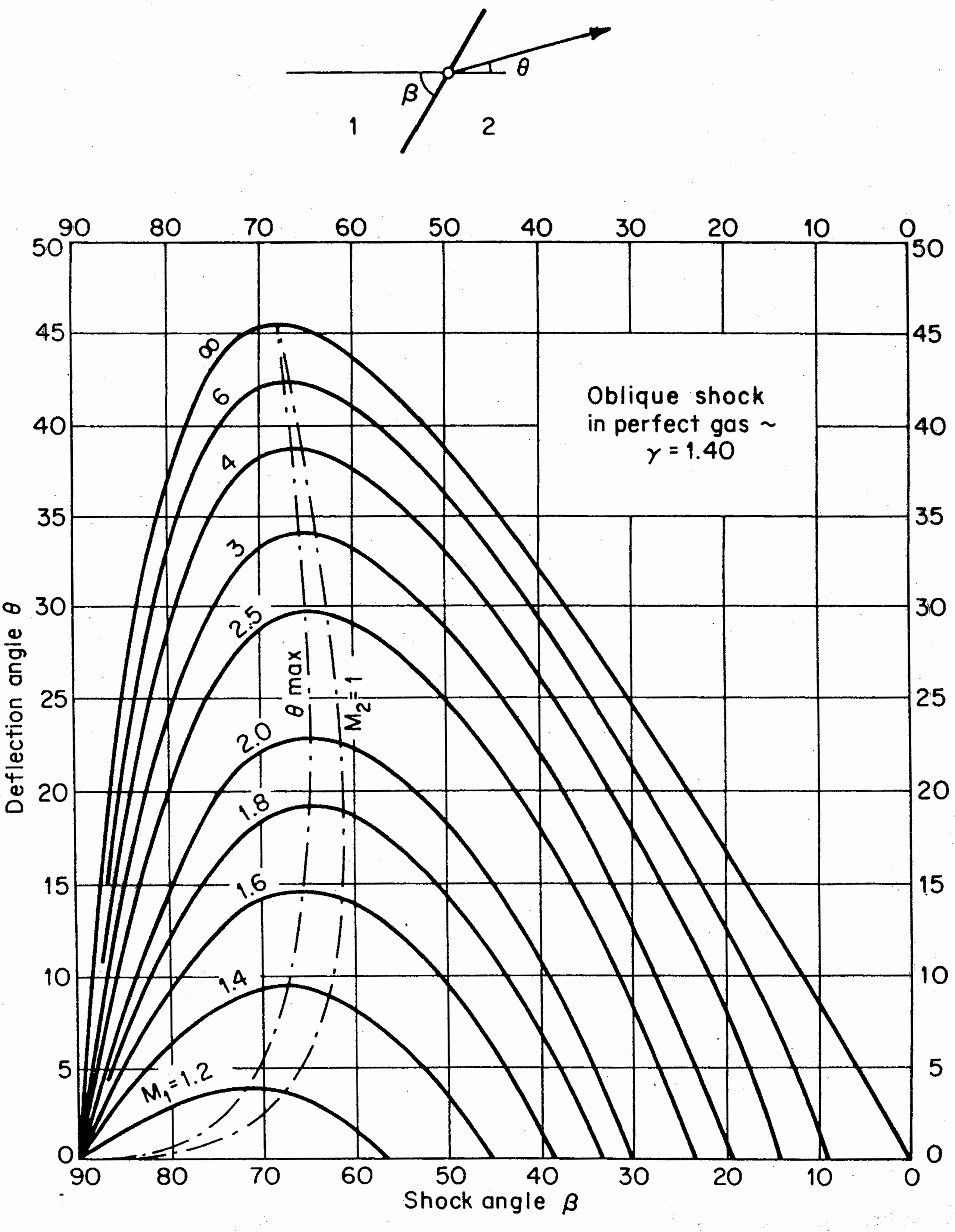
Show plot: (summarized in table D.3)

Note: in general $\beta(\theta)$ is double-valued! I.e. given θ there are two possible shock $\angle$'s β. Which is realized?

Shock-polar relations (another way to graphically visualize two solutions).

Lots of algebra so we will summarize key steps.

Goal: find $U_{2y} = f(U_{2x})$ $\Rightarrow$ visualize two solutions.



θ
β
1
2
Oblique shock
in perfect gas ~
γ = 1.40
Deflection angle θ
Shock angle β
θ max
$M_2 = 1$
$M_1 = 1.2$
1.4
1.6
1.8
2.0
2.5
3
4
6
∞

Figure D.1

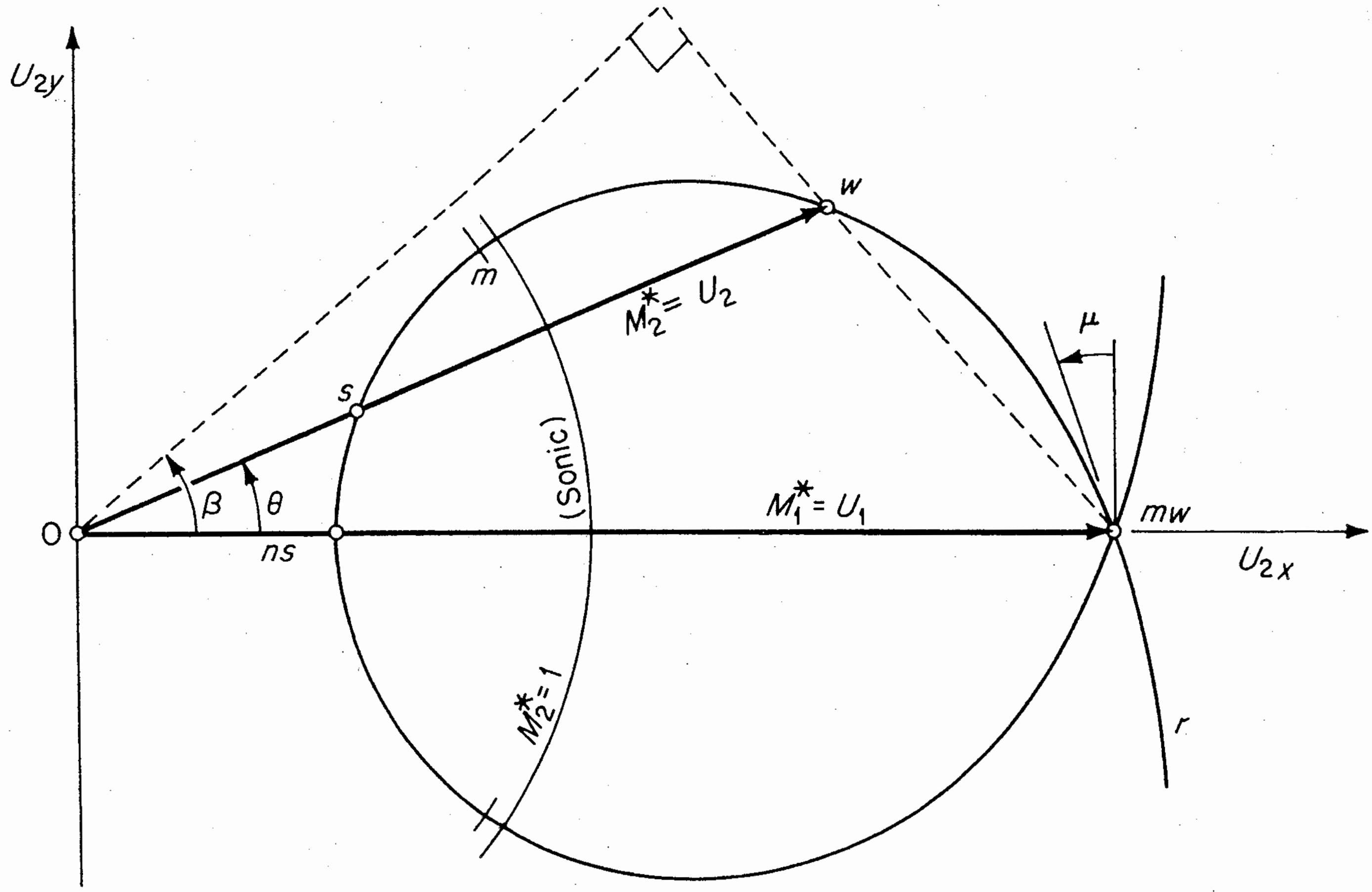
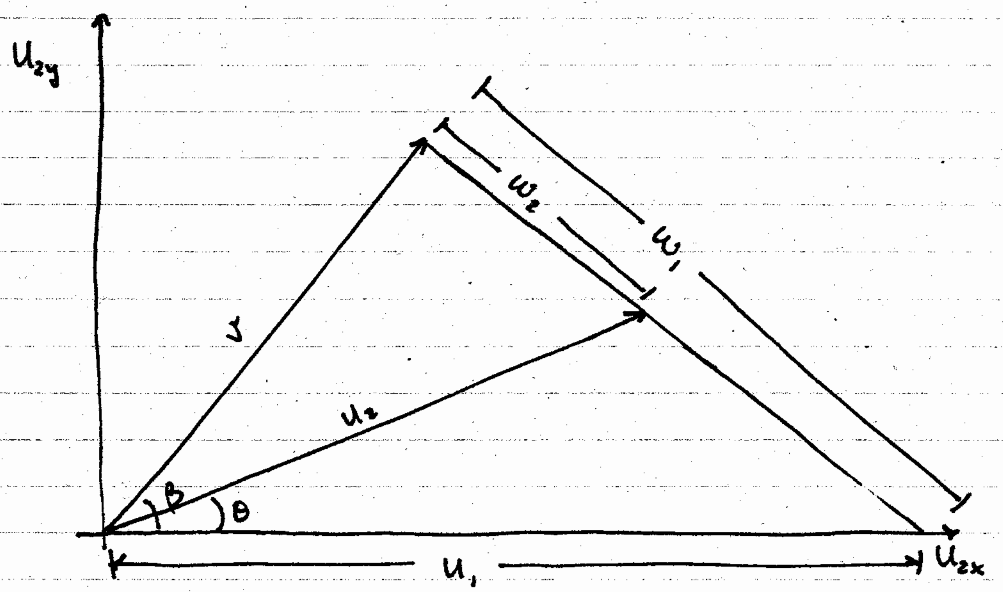


Figure 7.20
Oblique-shock polar diagram for the particular case $U_1 = M_1^* = 2$, $\gamma = 1.40$.

For any given turning angle θ, there are two possible distinc tions, which are conventionally called respectively the *weak solutic* the *strong solution*. It should be made clear that this nomenclatu



Start. w. cons. energy for perfect gas

$$u_1^2 + \frac{2}{\gamma-1} c_1^2 = u_2^2 + \frac{2}{\gamma-1} c_2^2$$

Using shock relations + much algebra (pg 329-330)

$$\boxed{w_1 w_2 = c_*^2 - \frac{\gamma-1}{\gamma+1} v^2}$$ Prandtl relation

Combine this w. geometric relations from Δ's above.
After more algebra:

$$\boxed{U_{2y}^2 = \frac{(U_1 - U_{2x})^2 (U_1 U_{2x} - 1)}{\frac{2}{\gamma+1} U_1^2 - U_1 U_{2x} + 1}}$$ where $U \equiv u/c_*$

Can find U_{2y} as a fn. of U_{2x} w. U_1 as a parameter

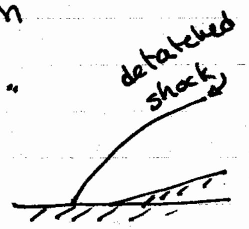
Show plot on O.H.

- For a given θ, there are two solutions, w and s. These are weak and strong solutions (Here "weak" and "strong" do <u>NOT</u> correspond to $\Pi \ll 1$ and $\Pi \gg 1$)

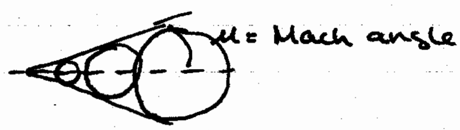
- ns = normal shock (note, u_1 and u_2 are parallel)

- If θ becomes too large $\rightarrow$ no solution

  $\Rightarrow$ shock becomes "detatched"



- mw = mach wave $(M_1^* = M_2^*)$



- $[w]$ = distance from mw to end of u_2^* vect

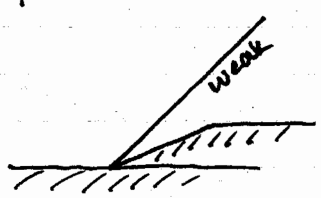
  $|[P]| = |-M_{1n}\, \rho_1 c_1 [w]|$

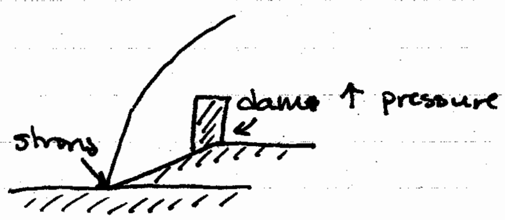
  $\Rightarrow$ $[P]_{strong} > [P]_{weak}$

- Branch r $\rightarrow$ rarefaction $\rightarrow$ decreases entropy $\rightarrow$ unphysical

<u>Which is observed physically?</u>

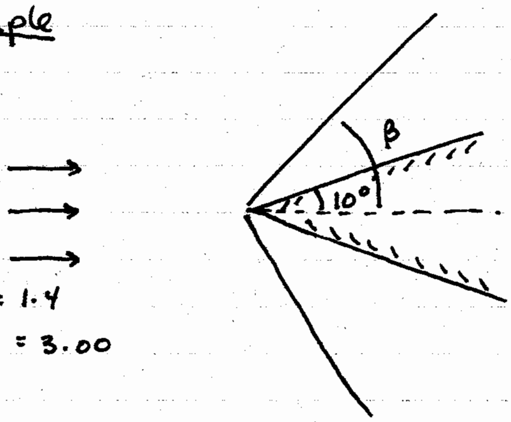
If downstream b.c.'s allow it, weak solution is observed (since this solution minimizes entropy production).





(higher dam $\rightarrow$ detatched)

Example

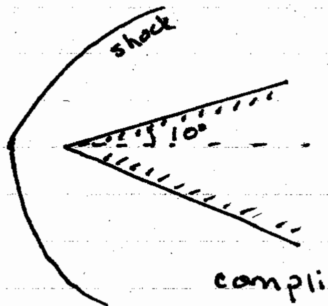


$\gamma = 1.4$
$M_1 = 3.00$

From tables : $\boxed{\beta = 27.38}$ $\quad p_2/p_1 = 2.055 \quad M_2 = 2.505$

Suppose $M_1 = 1.30$

From table : max $\theta = 6.67 \Rightarrow$ shock is detached



complicated downstream flow field
(detailed sol'n of eq. of motion
required)

Table D.3 Oblique Shock in a Perfect Gas ($\gamma = 1.40$) (*Continued*)

M_1	θ, degrees†	Weak Solutions			Strong Solutions		
		β, degrees	P_2/P_1	M_2	β	P_2/P_1	M_2
3.00	0.0	19.47	1.000	3.000	90.00	10.333	0.475
	2.0	20.87	1.166	2.898	89.30	10.322	0.476
	4.0	22.36	1.352	2.799	88.60	10.327	0.477
	6.0	23.94	1.562	2.701	87.88	10.319	0.480
	8.0	25.61	1.795	2.603	87.16	10.307	0.484
	10.0	27.38	2.055	2.505	86.41	10.292	0.489
	12.0	29.25	2.340	2.406	85.64	10.273	0.496
	14.0	31.22	2.654	2.306	84.84	10.248	0.504
	16.0	33.29	2.996	2.204	84.00	10.218	0.514
	18.0	35.47	3.368	2.100	83.11	10.182	0.525
	20.0	37.76	3.771	1.994	82.15	10.137	0.539
	22.0	40.19	4.206	1.886	81.11	10.082	0.556
	24.0	42.78	4.676	1.774	79.96	10.014	0.577
	26.0	45.55	5.184	1.659	78.65	9.927	0.602
	28.0	48.59	5.739	1.537	77.13	9.812	0.635
	30.0	52.02	6.356	1.406	75.24	9.652	0.678
	32.0	56.18	7.081	1.254	72.65	9.399	0.743
	34.0	63.67	8.268	1.003	66.75	8.697	0.908
	(34.07)	65.24	8.492	0.954	65.24	8.492	0.954
3.10	0.0	18.82	1.000	3.100	90.00	11.045	0.470
	2.0	20.21	1.171	2.994	89.32	11.043	0.470
	4.0	21.68	1.364	2.891	88.64	11.039	0.472
	6.0	23.26	1.582	2.789	87.95	11.031	0.474
	8.0	24.93	1.825	2.688	87.24	11.019	0.478
	10.0	26.69	2.096	2.586	86.52	11.004	0.483
	12.0	28.55	2.395	2.484	85.78	10.984	0.490
	14.0	30.51	2.724	2.380	85.00	10.960	0.497
	16.0	32.57	3.083	2.274	84.19	10.930	0.507
	18.0	34.74	3.474	2.167	83.33	10.894	0.518
	20.0	37.02	3.897	2.058	82.42	10.850	0.531
	22.0	39.42	4.354	1.947	81.42	10.795	0.548
	24.0	41.97	4.847	1.833	80.33	10.728	0.567
	26.0	44.69	5.379	1.715	79.09	10.644	0.591
	28.0	47.65	5.956	1.593	77.67	10.533	0.621
	30.0	50.94	6.592	1.462	75.94	10.383	0.661

† Figures in parentheses are maximum values.

M_1	degı
	32
	34
	(34
3.20	0.
	2.
	4.
	6.
	8.
	10.
	12.
	14.
	16.
	18.
	20.
	22.
	24.
	26.
	28.
	30.
	32.
	34.
	(35.
3.30	0.0
	2.0
	4.0
	6.0
	8.0
	10.0
	12.0
	14.0
	16.0
	18.0
	20.0
	22.0
	24.0
	26.0

† Figures in par