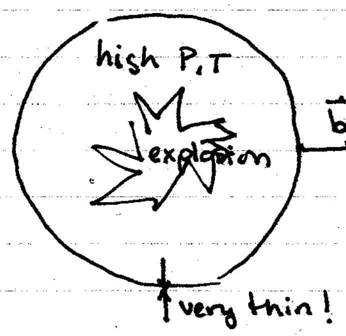
Shock waves

" A shock wave is a relatively thin region of rapid state variation across which there is a flow of matter. "

e.g.

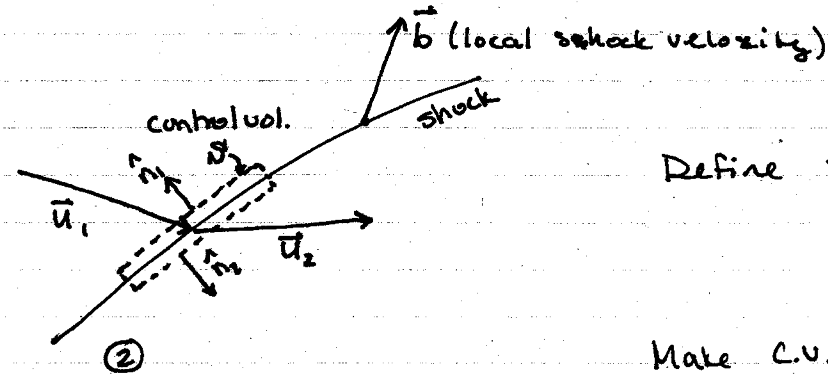


Idealize this as a discontinuity. All fluid properties (density, pressure, velocity, etc.) are discontinuous across this surface.

(show traffic pic. again. Note, traffic shock is not stationary).

Note: "real" shocks have finite (but very small!) thickness. Viscosity tends to smear the shock

Conservation laws across a shock



Define side 1 as inflow
side 2 as outflow
$(\hat{u}_2 - b) \cdot \hat{n}_2 > 0$

Make C.V. arbitrarily thin

Define: $w_1 \equiv -(\bar{u}_1 - \bar{b}) \cdot \hat{n}_1$

$$w_2 \equiv (\bar{u}_2 - \bar{b}) \cdot \hat{n}_2$$

<u>Mass</u>: $\cancel{\frac{d}{dt}\int_V \rho \, dV} + \int_S \rho (\bar{u} - \bar{b}) \cdot \hat{n} \, dA = 0$

make C.V. arbitrarily thin s.t. all $\int_V dV \to 0$

$$\rho_1 \underbrace{(\bar{u}_1 - \bar{b}) \cdot \hat{n}_1}_{-w_1} + \rho_2 \underbrace{(\bar{u}_2 - \bar{b}) \cdot \hat{n}_2}_{w_2} = 0$$

$$\Rightarrow \rho_2 w_2 - \rho_1 w_1 = 0 \quad \Rightarrow \quad \boxed{[\rho w] = 0}$$

Bracket notation: $[P] \equiv P_2 - P_1$

<u>Momentum</u>:

$$\cancel{\frac{d}{dt}\int_V \rho \bar{u} \, dV} + \int_S \rho \bar{u} (\bar{u} - \bar{b}) \cdot \hat{n} \, dA = \cancel{\int_V \rho \bar{G} \, dV} + \int_S \bar{T} \, dA$$

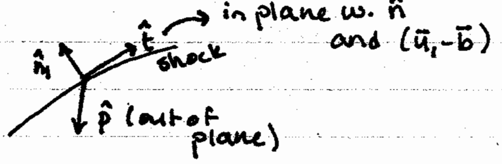
$$\rho_1 \bar{u}_1 \underbrace{(\bar{u}_1 - \bar{b}) \cdot \hat{n}_1}_{-w_1} + \rho_2 \bar{u}_2 \underbrace{(\bar{u}_2 - \bar{b}) \cdot \hat{n}_2}_{w_2} = -p_1 \hat{n}_1 - p_2 \hat{n}_2 \quad \text{(neglect visc. stress)}$$

$$\rho_2 w_2 \bar{u}_2 - \rho_1 w_1 \bar{u}_1 = -p_1 \hat{n}_1 - p_2 \hat{n}_2$$

To simplify, add $\bar{b}(\rho_1 w_1 - \rho_2 w_2)$ $(=0)$; note $\hat{n}_2 = -\hat{n}_1$

$$\rho_2 w_2 (\bar{u}_2 - \bar{b}) - \rho_1 w_1 (\bar{u}_1 - \bar{b}) = \hat{n}_1 (P_2 - P_1) \qquad (*)$$

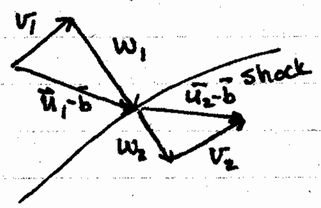
Choose basis:



Take dot product of $(*)$ w. each basis vector:

$$\rho_2 w_2 \underbrace{(\bar u_2-\vec b)\cdot(-\hat n_2)}_{-w_2} - \rho_1 w_1 \underbrace{(\vec u_1-\vec b)\cdot\hat n_1}_{-w_1} = P_2 - P_1$$

$$\boxed{[p + \rho w^2] = 0}$$



$$\rho_2 w_2 v_2 - \rho_1 w_1 v_1 = 0$$

$$\boxed{[v] = 0}$$ (From cons. of mass, $J = \rho_2 w_2 = \rho_1 w_1$)

$$\rho_2 w_2 (\bar u_2 - \vec b)\cdot\hat p = 0$$

$\Rightarrow (\bar u_2 - \vec b)$ is in the same plane as $(\bar u_1 - \vec b)$ and $\hat n_1$.

<u>Entropy</u>

$$\frac{d}{dt}\cancelto{0}{\int_V \rho s\, dV} + \int_S \rho s(\bar u - \vec b)\cdot\hat n\, dA + \int_S \cancelto{\text{adiabatic}}{\frac{1}{T}\vec q\cdot\hat n\, dA} \geq 0$$

$$\rho_1 s_1 \underbrace{(\bar u_1 - \vec b)\cdot\hat n_1}_{-w_1} + \rho_2 s_2 \underbrace{(\bar u_2 - \vec b)\cdot\hat n_2}_{w_2} \geq 0$$

$$\rho_2 w_2 s_2 - \rho_1 w_1 s_1 \geq 0 \quad \Rightarrow \quad \boxed{[s] \geq 0}$$

<u>Energy</u>

$$\frac{d}{dt}\int_V \cancel{\rho\left(e + \frac{u^2}{2}\right)} dV + \int_S \rho\left(e + \frac{u^2}{2}\right)(\bar u - \vec b)\cdot\hat n\, dA$$

$$= \int_V \cancel{\rho \vec G\cdot\hat n}\, dV + \int_S \bar T\cdot\bar u\, dA - \int_S \cancelto{\text{adiabatic}}{\vec q}\cdot\hat n\, dA$$

$$\rho_1\left(e_1 + \frac{u_1^2}{2}\right)\underbrace{(\bar u_1 - \vec b)\cdot\hat n_1}_{-w_1} + \rho_2\left(e_2 + \frac{u_2^2}{2}\right)\underbrace{(\bar u_2 - \vec b)\cdot\hat n_2}_{w_2} =$$

$$- P_1\, \hat n_1\cdot\bar u_1 - P_2\, \hat n_2\cdot\bar u_2$$

$$\rho_1\Big(\underbrace{e_1 + P_1/\rho_1}_{h_1} + \frac{u_1^2}{2}\Big)(-w_1) + \rho_2\Big(\underbrace{e_2 + P_2/\rho_2}_{h_2} + \frac{u_2^2}{2}\Big) w_2 =$$

$$- P_1\Big(\cancel{\hat n_1\cdot\bar u_1} + \underset{\uparrow \atop +(\cancel{\vec u_1}-\vec b)\cdot\hat n_1}{w_1}\Big) - P_2\Big(\cancel{\hat n_2\cdot\bar u_2} - \underset{\uparrow \atop (\cancel{\vec u_2}-\vec b)\cdot\hat n_2}{w_2}\Big)$$

$$\rho_2\left(h_2+\frac{u_2^2}{2}\right)w_2-\rho_1 w_1\left(h_1+\frac{u_1^2}{2}\right)=\underbrace{-P_1\,\vec{b}\cdot\hat{n}_1-P_2\,\vec{b}\cdot\hat{n}_2}_{(P_2-P_1)\,\vec{b}\cdot\hat{n}_1}$$

$$\cancel{\rho w}\left[h+\frac{u_2}{2}\right]=\left[\rho_2 w_2(\vec{u}_2-\vec{b})-\rho_1 w_1(\vec{u}_1-\vec{b})\right]\cdot\vec{b}$$

$$=\cancel{\rho w}\,(\vec{u}_2-\vec{u}_1)\cdot\vec{b}$$

If $\vec{b}=0$ (stationary shock), stagnation enthalpy is invariant across the shock.

$$(\vec{u}-\vec{b})^2=w^2+v^2 \Rightarrow u^2-2\vec{u}\cdot\vec{b}+b^2=w^2+v^2$$

$$\vec{u}\cdot\vec{b}=-\tfrac{1}{2}\left(w^2+v^2-u^2-b^2\right)$$

$$h_2+\frac{u_2^2}{2}-h_1-\frac{u_1^2}{2}=-\tfrac{1}{2}\left(w_2^2+\cancel{v_2^2}-\cancel{u_2^2}-\cancel{b^2}\right)$$
$$+\tfrac{1}{2}\left(w_1^2+\cancel{v_1^2}-\cancel{u_1^2}-\cancel{b^2}\right)$$

$$\boxed{\left[h+\tfrac{1}{2}w^2\right]=0}$$

<u>Properties of shock waves</u>

$$[P]+[\rho w^2]=0$$

$$[P]+\left[J^2\frac{1}{\rho}\right]=0 \quad\Rightarrow\quad J^2=\frac{-[P]}{[v]}=\frac{J[w]}{[v]}$$

Define shock Mach # : $M_{1n}\equiv\dfrac{w_1}{c_1}$

$\uparrow$ side 1 normal to shock

$$[P]+J[w]=0$$

normalize $\longrightarrow$

$$\frac{[P]}{\rho_1 c_1^2}=\frac{-\cancel{\rho_1} w_1[w]}{\cancel{\rho_1}\,c_1^2}=-M_{1n}\frac{[w]}{c_1}=-M_{1n}\frac{J[v]w_1}{c_1\;w_1}=-M_{1n}^2\frac{[v]}{v_1}$$

$\equiv \Pi$ dimensionless ~~normalized~~ pressure

$\Pi \ll 1$ weak shock $\Rightarrow$ $[P]$ is small compared to pressure ahead of shock.

$\Pi \gg 1$ strong shock (Note: $\rho_1 c_1^2 = \gamma P_1$ for perfect fluids)

$$[h + \tfrac{1}{2} w^2] = 0$$

$$[h] + \tfrac{1}{2}(w_2^2 - w_1^2) = 0$$

$$[h] + \tfrac{1}{2}(J^2 v_2^2 - J^2 v_1^2) = 0$$

$$[h] + \tfrac{1}{2}\left(-\frac{[P]}{[v]}\right)(v_2^2 - v_1^2) = 0$$

$$[h] - \tfrac{1}{2}[P]\frac{(v_2+v_1)\cancel{(v_2-v_1)}}{\cancel{(v_2-v_1)}} = 0$$

$$\boxed{h_2 - h_1 = \tfrac{1}{2}(P_2 - P_1)(v_2 + v_1)}$$ Rankine-Hugoniot equation

Note: contains ONLY thermodynamic quantities

P_1 and v_1 are given $\Rightarrow$ $h_2 = h_2(P_2, v_2)$

Combine w. eq. of state $\Rightarrow$ $\boxed{P_2 = P_2(v_2)}$ shock adiabat

For a perfect gas:

$h = c_p T + \text{const.}$ $\qquad$ $Pv = RT$

$$\Rightarrow h = \frac{\gamma P v}{\gamma - 1}$$

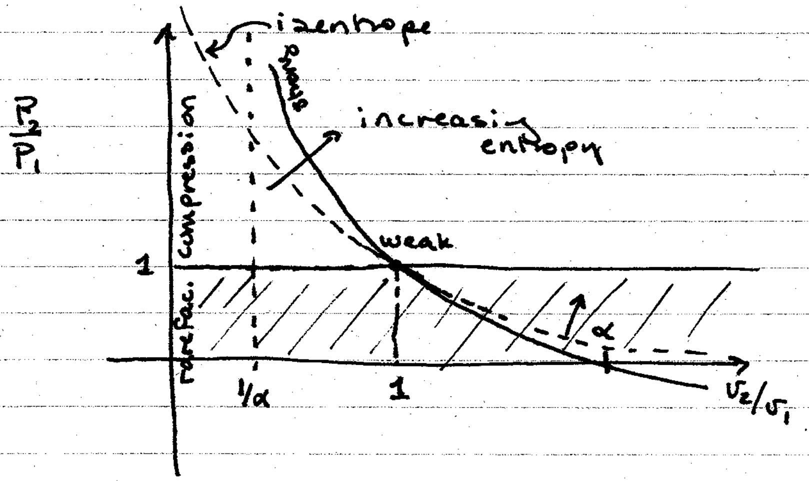
$$\frac{\gamma}{\gamma-1}(P_2 v_2 - P_1 v_1) = \tfrac{1}{2}(P_2 - P_1)(v_2 + v_1)$$

Rearrange:

$$\boxed{\frac{P_2}{P_1} = \frac{\frac{\gamma+1}{\gamma-1} - \frac{v_2}{v_1}}{\frac{\gamma+1}{\gamma-1}\frac{v_2}{v_1} - 1}}$$ Shock-adiabat curve for a perfect gas

$[P] > 0 \Rightarrow$ compression shock

$[P] < 0 \Rightarrow$ rarefaction shock



let $\frac{\gamma+1}{\gamma-1} \equiv \alpha$

$\frac{v_2}{v_1} = \alpha \Rightarrow P_2/P_1 = 0$

$\frac{v_2}{v_1} = \frac{1}{\alpha} \Rightarrow P_2/P_1 \to \infty$ $(\Rightarrow \Pi \gg 1)$

$\frac{v_2}{v_1} = 1 \Rightarrow P_2/P_1 = 1$

Compare this w. an isentrope: $\frac{P_2}{P_1} = \left(\frac{v_2}{v_1}\right)^{-\gamma}$

Since $[s] \geq 0$, only compression shocks are allowed!

@ $P_2/P_1 = 1 \Rightarrow P_2 = P_1$; $v_2 = v_1 \Rightarrow$ no shock!

These qualities are true in general, not just for a perfect fluid. We can show this holds analytically for any fluid in the limit of a weak shock.

$$[h] = \tfrac{1}{2}(P_2 - P_1)(v_2 + v_1) = \tfrac{1}{2}[P][v_2 - v_1 + 2v_1]$$

$$[h] = \tfrac{1}{2}[P][v] + v_1[P]$$

Expand in Taylor series about $P_2/P_1 = 1$ in P and s.

$$h_1 = h(s_1, P_1) \qquad v_1 = v(s_1, P_1)$$

$$h_2 = h(s_1 + \Delta s, P_1 + \Delta P) \qquad v_2 = v(s_1 + \Delta s, P_1 + \Delta P)$$

$$\Rightarrow \Delta s = [s]; \qquad \Delta P = [P]$$

$$[h] = \cancel{h_1} + \Delta s \overbrace{\left(\frac{\partial h}{\partial s}\right)_{p_1}}^{T_1} + \Delta P \overbrace{\left(\frac{\partial h}{\partial P}\right)_{s_1}}^{v_1} + \cancel{\tfrac{1}{2}\Delta s^2 \left(\frac{\partial^2 h}{\partial s^2}\right)_{p_1}} \ldots - \cancel{h_1}$$

$$[v] = \cancel{v_1} + \Delta s \left(\frac{\partial v}{\partial s}\right)_{p_1} + \Delta P \left(\frac{\partial v}{\partial P}\right)_{s_1} + \ldots - \cancel{v_1}$$

$$\Delta s\, T_1 + \cancel{\Delta P\, v_1} = v_1 \cancel{\Delta P} + \underbrace{\tfrac{1}{2}\Delta P \left\{ \Delta s\right.}_{\text{second order so we need to include 2nd order in } [h] \ldots} \left. \left(\frac{\partial v}{\partial s}\right)_{p_1} + \Delta P \left(\frac{\partial v}{\partial P}\right)_{s_1} + \ldots \right\}$$

$$\Delta s\, T_1 + \tfrac{1}{2}\Delta s^2 \left(\frac{\partial^2 h}{\partial s^2}\right)_{p_1} + \tfrac{1}{2}\Delta P^2 \left(\frac{\partial^2 h}{\partial P^2}\right)_{s_1} + \Delta s \Delta P \left(\frac{\partial^2 h}{\partial P \partial s}\right) + \ldots$$

$$\Delta s\, T_1 + \tfrac{1}{2}\Delta s^2 \left(\frac{\partial T_1}{\partial s}\right)_{p_1} + \tfrac{1}{2} \cancel{\Delta P^2 \left(\frac{\partial v}{\partial P}\right)_{s_1}} + \Delta s \Delta P \left(\frac{\partial T}{\partial P}\right) =$$

$$\tfrac{1}{2}\Delta P\, \Delta s \left(\frac{\partial v}{\partial s}\right)_{p_1} + \cancel{\tfrac{1}{2}\Delta P^2 \left(\frac{\partial v}{\partial P}\right)_{s_1}}$$

$$\cancel{\Delta s T_1} + \tfrac{1}{2}\Delta s\,(T_2 - \cancel{T_1}) + \tfrac{1}{2}\Delta s\,(T_2 - \cancel{T_1}) = 0 \quad \Rightarrow \boxed{\Delta s = 0}$$

So we need to go to next order!

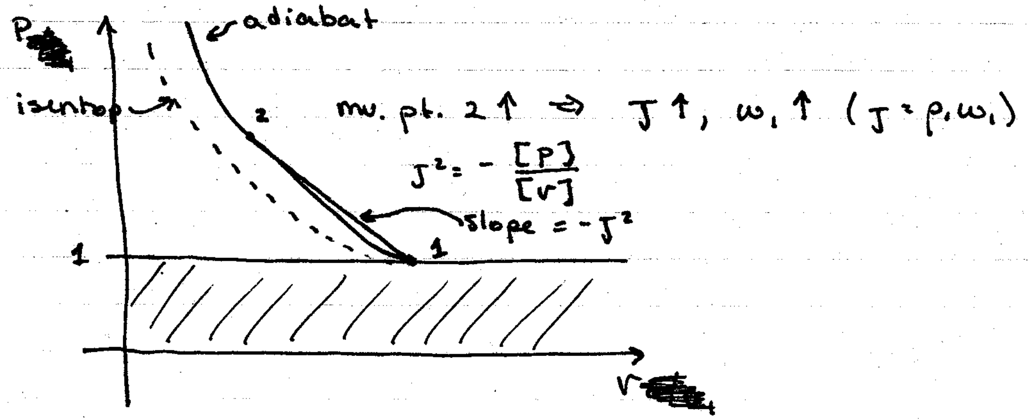
To $\Theta([P]^3)$:

$$[s] = \frac{1}{12 T_1} \underbrace{\left(\frac{\partial^2 v}{\partial P^2}\right)_s}_{>0 \text{ for all normal fluids}} [P]^3 + \Theta([P]^4)$$

$\Rightarrow \quad [P] > 0 \quad (\text{since } [s] \geq 0) \quad \therefore$ only compression shocks allowed

Also note, as $[P] \to 0$, $[s] \to 0$ VERY rapidly.
$\therefore$ weak shocks are approx isentropic.

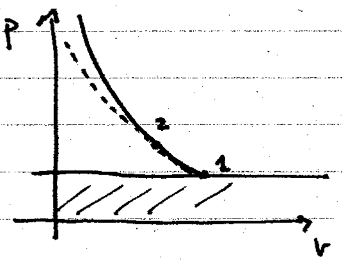
Fix ρ_1, P_1



Isentrope has slope:

$$\left(\frac{\partial P}{\partial v}\right)_s = \frac{-c^2}{v^2} = -\rho^2 c^2$$

by def'n of speed of sound



$$\left(\frac{\partial P}{\partial v}\right)_{s(2)} \leq -J^2 \leq \left(\frac{\partial P}{\partial v}\right)_{s(1)}$$

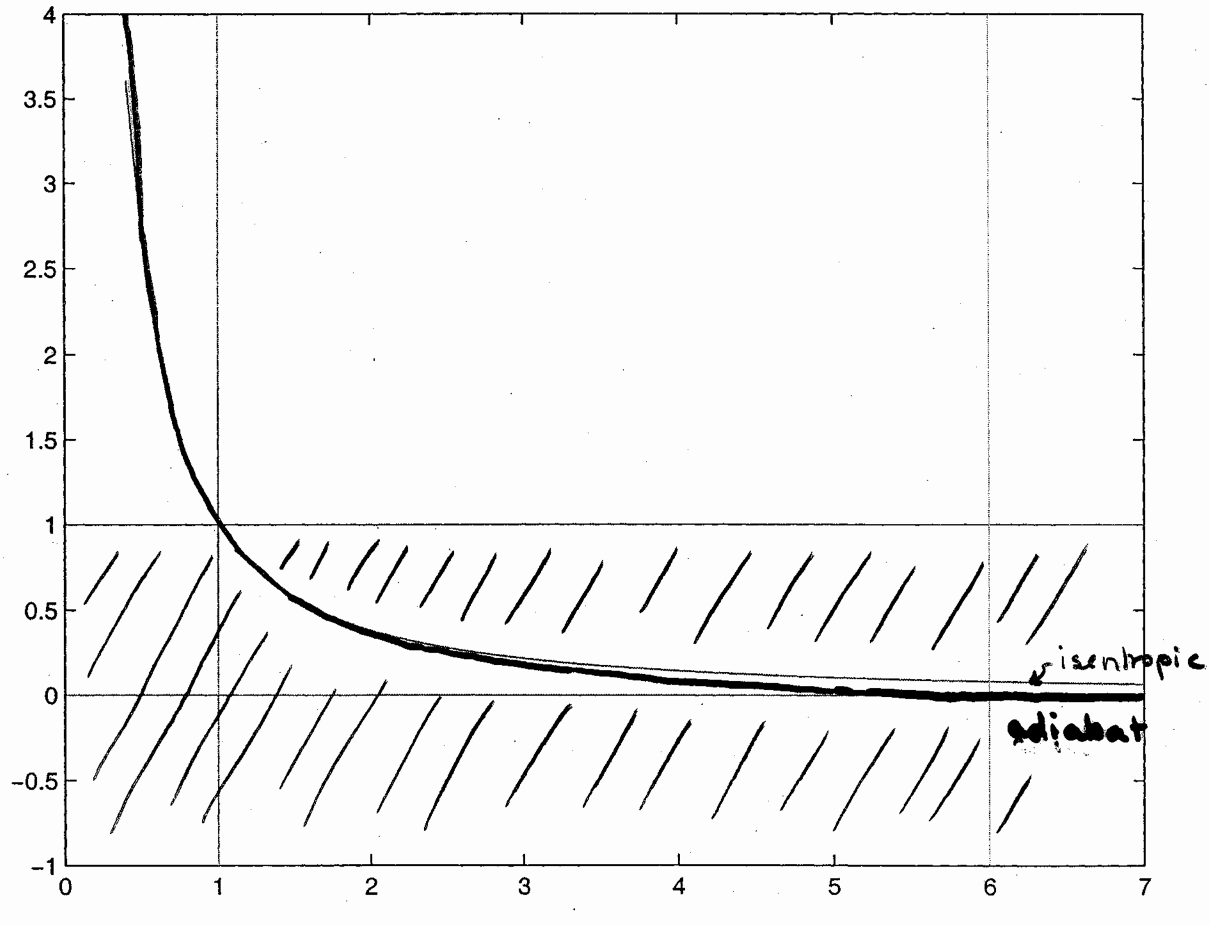
$$\rho_2^2 c_2^2 \geq \underset{\substack{\shortparallel \\ \rho_2^2 \omega_2^2 = \rho_1^2 \omega_1^2}}{J^2} \geq \rho_1^2 \omega_1^2$$

$\omega_1 \geq c_1$ ←	upstream vel. is supersonic (relative to shock vel.)
$\omega_2 \leq c_2$ ←	downstream is subsonic

∴ downstream disturbance has no effect on upstream flow

(Much of this holds for strong shocks as well but proof is more complicated. PROJECT)

Limit of very weak shock, $\omega_1 \to c_1$; $\omega_2 \to c_2$, isentropic ⇒ sound waves!



4
3.5
3
2.5
2
1.5
1
0.5
0
-0.5
-1
0
1
2
3
4
5
6
7
isentropic
adiabat

	Type of plane	Top Speed	First Flight	Current Status	Replacement ?
SR-71 "Blackbird" (US)	Reconnaissance	Mach 3.2 (cruising) LA-DC 1hr 4min	12/2/1964	Retired 1990 (a few inservice until 1998)	NONE (but lots of conspiracy theories)
Concorde (Britain/France)	Commercial	Mach 2.02 (cruising) ATW in 31hrs 27 min	3/2/1969 (prototype) 1/21/1976 (in service)	Retired 5/31/2003	NONE (collab btwn EADS-Japan? Paris-Tokyo in 2 hrs?)
MiG-25 "Foxbat" (Russian)	Fighter (?)	~ Mach 3 (?)	1964	?	?
F-15 (US)	Fighter	Mach 2.5	1972 (1st flight) 1976 (in service)	Begin to phase out starting 2005 (?)	F-22 "Raptor" Mach 1.5 (cruising)