The Equations of Fluid Mechanics

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Boundary Layer Equations ($\delta(x) << x$)

$$\rho\left(v_x\frac{\partial v_x}{\partial x}+v_y\frac{\partial v_x}{\partial y}\right)=-\frac{\partial p}{\partial x}+\mu\frac{\partial^2 v_x}{\partial y^2}$$

$$0=-\frac{\partial p}{\partial y} \qquad \left(\frac{\partial v_x}{\partial x}+\frac{\partial v_y}{\partial y}\right)=0$$

Reynolds Transport Theorem $B_{cv}=\int_{cv}\phi dV=\int_{cv}\rho b\, dV$

Form A: $$\frac{dB_{sys}}{dt}=\frac{d}{dt}\int_{cv}\phi dV+\int_{CS}(\rho b)\left(\boldsymbol{v}-\boldsymbol{v}_c\right)\cdot\mathbf{n}\, dA$$

Form B: $$\frac{dB_{sys}}{dt}=\int\frac{\partial\phi}{\partial t}dV+\int_{CS}(\rho b)\boldsymbol{v}\cdot\mathbf{n}\, dA$$

Cauchy Momentum Equation

$$\rho\frac{D\boldsymbol{v}}{Dt}=-\nabla p+\nabla\cdot\boldsymbol{\tau}+\rho\mathbf{g}$$

Continuity

$$\frac{1}{\rho}\frac{D\rho}{Dt}=-(\nabla\cdot\boldsymbol{v})$$

Young-Laplace Equation

$$\Delta p=\sigma\left(\frac{1}{R_1}+\frac{1}{R_2}\right)$$

Free Surface Flows; $p\approx-\sigma\left(\partial^2 h/\partial x^2\right)$

$$\frac{\partial h}{\partial t}+\frac{\partial Q'}{\partial x}=0 \quad \text{where } Q'=\int_0^h v_x\, dy$$

$$\frac{\partial h}{\partial t}+\left[\frac{\rho g h^2\sin\theta}{\mu}\right]\frac{\partial h}{\partial x}=\frac{\rho g\cos\theta}{3\mu}\frac{\partial}{\partial x}\left(h^3\frac{\partial h}{\partial x}\right)$$

Newtonian Constitutive Equation

$$\boldsymbol{\tau}=\mu\left\{\nabla\boldsymbol{v}+(\nabla\boldsymbol{v})^t\right\}=\mu\dot{\boldsymbol{\gamma}}$$

Incompressible Flow

$$\nabla\cdot\boldsymbol{v}=0$$

Quasi-Fully Developed Flow (Lubrication approximation)

$$h/L<<1,\; Re_L\left(h/L\right)^2<<1,\; h^2/(\nu\tau)<<1$$

$$0\approx-\frac{\partial p}{\partial x}+\mu\frac{\partial^2 v_x}{\partial y^2}+\rho g_x \qquad 0\approx\frac{\partial p}{\partial y}+\rho g_y$$

Navier-Stokes Equation for isothermal flow of an incompressible Newtonian fluid

$$\rho\frac{D\boldsymbol{v}}{Dt}=\rho\left\{\frac{\partial\boldsymbol{v}}{\partial t}+\boldsymbol{v}\cdot\nabla\boldsymbol{v}\right\}=-\nabla p+\mu\nabla^2\boldsymbol{v}+\rho\mathbf{g}$$

Hydrostatics

$$\nabla p=\rho(\mathbf{g}-\mathbf{a})$$

Very viscous fluid (Stokes' Flow; $Re\to 0$**)**

$$\mathbf{0}=-\nabla p+\mu\nabla^2\boldsymbol{v}+\rho\mathbf{g}$$

Inviscid fluid $Re\to\infty$ **(Euler Equation)**

$$\rho\frac{D\boldsymbol{v}}{Dt}=-\nabla p+\rho\mathbf{g}$$

Across streamlines (outward pointing normal $\boldsymbol{n}$)

$$\frac{\partial p}{\partial n}=\frac{\rho v^2}{\mathcal{R}}$$

Steady Irrotational Flow ($\boldsymbol{\omega}=0$)
Velocity potential $\boldsymbol{v}=\nabla\phi$

$$(\nabla\cdot\boldsymbol{v})=\nabla^2\phi=0$$

$\boldsymbol{\omega}=0$

Unsteady Bernoulli Equation *along a streamline*

$$\int_1^2\frac{\partial\boldsymbol{v}}{\partial t}\cdot d\mathbf{r}+\left(\tfrac{1}{2}v_2^2-\tfrac{1}{2}v_1^2\right)+\int_1^2\frac{dp}{\rho}+\left(\Pi_2-\Pi_1\right)=0$$

Kutta-Zhukowski Theorem

$$\Gamma=\int\boldsymbol{\omega}\cdot d\mathbf{A}=\oint\boldsymbol{v}\cdot d\mathbf{s}$$

$$L=\rho V\Gamma \qquad D=0$$

Steady Bernoulli Equation along a streamline for an inviscid flow of an incompressible fluid

$$\left(\tfrac{1}{2}v_2^2-\tfrac{1}{2}v_1^2\right)+\frac{1}{\rho}\left(p_2-p_1\right)+g\left(z_2-z_1\right)=0$$