13.42 HW # 2 SOLUTIONS

1 a. GIVEN : $y(t) = \int_0^{t+\alpha} u(s)\, ds$

- $\int_0^{t+\alpha} [a_1 u_1(s) + a_2 u_2(s)]\, ds = a_1 \int_0^{t+\alpha} u_1(s)\, ds + a_2 \int_0^{t+\alpha} u_2(s)\, ds$

$$= a_1 y_1(t) + a_2 y_2(t)$$

$\therefore$ LINEAR

- 1.) RHS: $\int_0^{t+\alpha} u(s+\tau)\, ds$

$$\left[\begin{array}{ll} \text{LET } \xi = s+\tau & s = t+\alpha \longrightarrow \xi = t+\alpha+\tau \\ d\xi = ds & s = 0 \longrightarrow \xi = \tau \end{array} \right.$$

$$= \int_\tau^{t+\alpha+\tau} u(\xi)\, d\xi$$

2.) LHS: $y(t+\tau) = \int_0^{t+\tau+\alpha} u(s)\, ds$

RHS $\neq$ LHS, $\therefore$ NOT T.I.

b. GIVEN : $y(t) = \int_{t-\alpha}^{t+\alpha} [u(s)]^2\, ds$

- $\int_{t-\alpha}^{t+\alpha} [a_1 u_1(s) + a_2 u_2(s)]^2\, ds \neq a_1 \int_{t-\alpha}^{t+\alpha} [u_1(s)]^2\, ds +$

$$+ a_2 \int_{t-\alpha}^{t+\alpha} [u_2(s)]^2\, ds$$

$\therefore$ NONLINEAR

- $\int_{t-\alpha}^{t+\alpha} [u(s+\tau)]^2\, ds = \int_{t-\alpha+\tau}^{t+\alpha+\tau} [u(\xi)]^2\, d\xi = y(t+\tau)$

$\therefore$ TIME INVARIANT.

c. GIVEN: $y(t) = \alpha \frac{du}{dt}(t) \left| \frac{du}{dt}(t) \right|$

- $\alpha \left(a_1 \frac{du_1}{dt} + a_2 \frac{du_2}{dt} \right) \left| a_1 \frac{du_1}{dt} + a_2 \frac{du_2}{dt} \right| \neq$

$$a_1 \alpha \frac{du_1}{dt} \left| \frac{du_1}{dt} \right| + a_2 \alpha \frac{du_2}{dt} \left| \frac{du_2}{dt} \right|$$

$\therefore$ NONLINEAR

- $\alpha \frac{du}{dt}(t+\tau) \left| \frac{du}{dt}(t+\tau) \right| = \alpha \frac{du}{dt}(t+\tau) \left| \frac{du}{dt}(t+\tau) \right|$

(RHS) (LHS)

$\therefore$ TIME INVARIANT

d. GIVEN: $\alpha \ddot{y}(t) + \beta \dot{y}(t) + \gamma y(t) = u(t)$

- $a_1 u_1 + a_2 u_2 = a_1 \left(\alpha \ddot{y}_1 + \beta \dot{y}_1 + \gamma y_1 \right) + a_2 \left(\alpha \ddot{y}_2 + \beta \dot{y}_2 + \gamma y_2 \right)$

$\therefore$ LINEAR

- $u(t+\tau) = \alpha \ddot{y}(t+\tau) + \beta \dot{y}(t+\tau) + \gamma y(t+\tau)$

(RHS) (LHS)

$\therefore$ TIME INVARIANT

2 a. $a_1 \cos \omega t \longrightarrow \boxed{\text{LTI}} \longrightarrow a_2 \cos(\omega t + \phi)$

b. $\sin 5t \longrightarrow \boxed{\text{NOT LTI}} \longrightarrow 2 \cos(10t + \pi)$

3a. $$\tilde{f}(\omega) = \int_{-\infty}^{\infty} f(t)\, e^{-i\omega t}\, dt$$

$$= \int_{-\infty}^{\infty} u_0(t - \tau)\, e^{-i\omega t}\, dt$$

$$= \underline{e^{-i\omega\tau}}$$

b. $$\int_{-\infty}^{\infty} \frac{df}{dx}\, e^{-i\alpha x}\, dx$$

$$\left[\begin{array}{ll} \text{LET } u = e^{-i\alpha x}, & dv = \frac{df}{dx}\, dx \\ du = -i\alpha e^{-i\alpha x}\, dx, & v = f \end{array} \right.$$

$$= \left[f e^{-i\alpha x} \right]_{-\infty}^{\infty} - \int_{-\infty}^{\infty} f(-i\alpha)\, e^{-i\alpha x}\, dx$$

$$= 0 + i\alpha \int_{-\infty}^{\infty} f e^{-i\alpha x}\, dx$$

$$= \underline{i\alpha \tilde{f}(\alpha)}$$

c. $$\int_{-\infty}^{\infty} \frac{d^2 f}{dx^2}\, e^{-i\alpha x}\, dx$$

$$\left[\begin{array}{ll} \text{LET } u = e^{-i\alpha x}, & dv = \frac{d^2 f}{dx^2}\, dx \\ du = -i\alpha e^{-i\alpha x}\, dx, & v = \frac{df}{dx} \end{array} \right.$$

$$= \left[\frac{df}{dx} e^{-i\alpha x} \right]_{-\infty}^{\infty} - \int_{-\infty}^{\infty} \frac{df}{dx} \left(-i\alpha e^{-i\alpha x}\right) dx$$

$$= 0 + i\alpha \underbrace{\int_{-\infty}^{\infty} \frac{df}{dx}\, e^{-i\alpha x}\, dx}_{= i\alpha \tilde{f}(\alpha)}$$

$$= \underline{(i\alpha)^2\, \tilde{f}(\alpha)}.$$

4 a. GIVEN: $m\ddot{x}(t) + c\dot{x}(t) + kx(t) = f(t)$

$$f(t) = \mathbb{R}e\left\{ \mathbb{F} e^{i\omega t} \right\}$$

$$x(t) = \mathbb{R}e\left\{ \mathbb{X} e^{i\omega t} \right\}$$

$$\longrightarrow \; m(-\omega^2)\mathbb{X}\cancel{e^{i\omega t}} + ci\omega\mathbb{X}\cancel{e^{i\omega t}} + k\mathbb{X}\cancel{e^{i\omega t}} = \mathbb{F}\cancel{e^{i\omega t}}$$

$$(-\omega^2 m + i\omega c + k)\,\mathbb{X} = \mathbb{F}$$

FROM READING 3, WE KNOW THAT, GIVEN

$$f(t) \longrightarrow \boxed{\text{LTI}} \longrightarrow x(t)$$

$$\mathbb{X}(\omega) = H(\omega)\,\mathbb{F}(\omega)$$

$$\therefore \; H(\omega) = \frac{1}{-\omega^2 m + i\omega c + k}$$

b. LET $f_1(t) = \mathbb{R}e\left\{ \mathbb{F}_1 e^{i\omega t} \right\}$ & $f_2(t) = \mathbb{R}e\left\{ \mathbb{F}_2 e^{i\omega t} \right\}$

THEN $f(t) = \alpha\,\mathbb{R}e\left\{ \mathbb{F}_1 e^{i\omega t} \right\} + \beta\,\mathbb{R}e\left\{ \mathbb{F}_2 e^{i\omega t} \right\}$

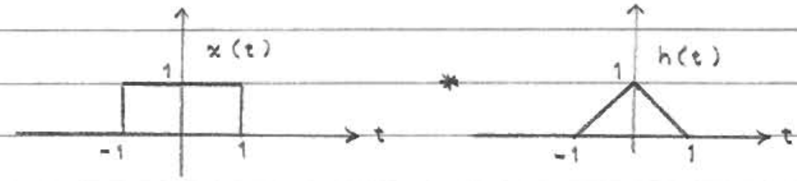
$$= \mathbb{R}e\left\{ \underbrace{(\alpha\mathbb{F}_1 + \beta\mathbb{F}_2)}_{\mathbb{F}} e^{i\omega t} \right\}$$

- $\mathbb{X} = H(\omega)\,\mathbb{F}$

$$= \frac{\alpha\mathbb{F}_1 + \beta\mathbb{F}_2}{-\omega^2 m + i\omega c + k}$$

AND $x(t) = \mathbb{R}e\left\{ \left(\dfrac{\alpha\mathbb{F}_1 + \beta\mathbb{F}_2}{-\omega^2 m + i\omega c + k} \right) e^{i\omega t} \right\}$.

5 a.



$$\boxed{\begin{aligned} y(t) &= x(t) * h(t) \\ &= \int_{-\infty}^{\infty} x(\tau)\, h(t-\tau)\, d\tau \end{aligned}}$$

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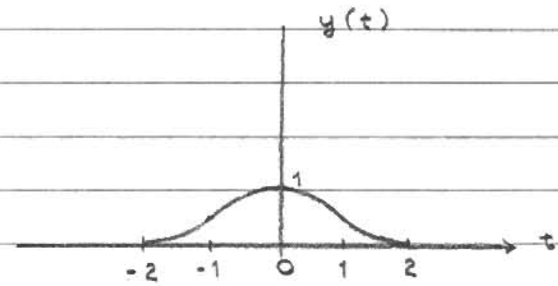
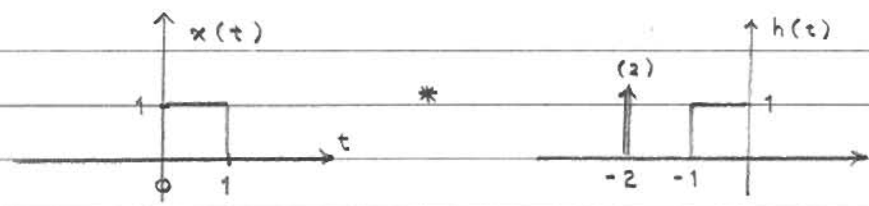


b.



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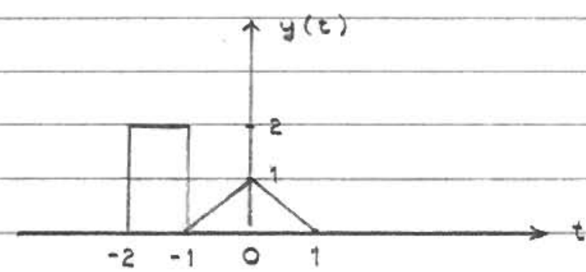

6a. LET $f(t) = f_o \cos(\omega_o t + \psi)$

$$= \mathbb{R}e\left\{ f_o e^{i(\omega_o t + \psi)} \right\}$$

$$= \mathbb{R}e\left\{ \mathbb{F}_o e^{i\omega_o t} \right\} \qquad \text{WHERE } \mathbb{F}_o = f_o e^{i\psi}$$

AND $y(t) = h(t) * f(t)$:

$$\bullet \quad y(t) = \int_{-\infty}^{\infty} h(\tau)\, f(t-\tau)\, d\tau$$

$$= \int_{-\infty}^{\infty} h(\tau)\, \mathbb{R}e\left\{ \mathbb{F}_o e^{i\omega_o(t-\tau)} \right\} d\tau$$

$$= \mathbb{R}e\left\{ \int_{-\infty}^{\infty} h(\tau)\, e^{-i\omega_o \tau}\, \mathbb{F}_o e^{i\omega_o t}\, d\tau \right\}$$

$$= \mathbb{R}e\left\{ H(\omega_o)\, \mathbb{F}_o e^{i\omega_o t} \right\}$$

LET $H(\omega_o) = |H(\omega_o)|\, e^{i\varphi}$

$$= \mathbb{R}e\left\{ |H(\omega_o)|\, f_o e^{i(\omega_o t + \psi + \varphi)} \right\}$$

$$= |H(\omega_o)|\, f_o \cos(\omega_o t + \psi + \varphi).$$